SOME ROTATION MODELS FOR MULTI-LOCATION, MULTI-GRADE PERSONNEL SYSTEMS

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THESIS

SOME ROTATION MODELS FOR MULTI-LOCATION, MULTI-GRADE PERSONNEL SYSTEMS

by

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September 1974

Dissertation Supervisor: K. T. Marshall

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Some Rotation Models for Multi-Location, Multi-Grade Personnel Systems		5. TYPE OF REPORT & PERIOD COVERED Doctoral Dissertation; September 1974
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Robert Dale Rantschler		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		12. REPORT DATE September 1974
		13. NUMBER OF PAGES 180
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Naval Postgraduate School Monterey, California 93940		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Rotation Models
Personnel Planning
Personnel Selection Process
Markov Flow Model
Manpower Planning
Manpower Budget
Manpower Policy Evaluation
Transportation Problem

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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(Page 1) S/N 0102-014-6601

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Block #20 continued

in one or more of these three factors are discussed, and the model is used to analyze the effects of such changes on the total transfer cost. Another model examines the problem of transferring personnel between locations and develops a parameterization of the personnel selection process when transfer costs are taken into consideration. The third model examines the problem encountered in searching for qualified people to replace those leaving specified positions. The models, although developed independently, are shown to interrelate on a macroscopic level.

Some Rotation Models for Multi-Location, Multi-Grade Personnel Systems

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DOCTOR OF PHILOSOPHY

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NAVAL POSTGRADUATE SCHOOL
September 1974

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ACKNOWLEDGMENTS

To Professor Kneale T. Marshall: Your expertise, guidance and insight were invaluable in arriving at this final work. Your encouragement of my efforts and your professional dedication kept spirits up during the long weeks of this past year.

To Professor Donald P. Gaver: Without your inspiration, I may never have made the effort. With your willingness to contribute, several segments of this work are presented in a better manner than would otherwise have been possible.

To Professors Gilbert T. Howard, Peter A. W. Lewis, Edward B. Thornton and Harold A. Titus: Your assistance in the review of the thesis is greatly appreciated. Your comments were found quite useful in the final preparation of the thesis.

I. INTRODUCTION

A. BACKGROUND

In the recent past, concern about the inability of large institutions with multi-location, multi-grade personnel systems to transfer personnel and remain within budgetary limitations has grown rapidly. Transfers within an institution are often required to maintain operations of the institution, to develop present personnel resources to meet forecasted needs of future operations, and to share the burden of "hardship" tours in locations of unusual economic, political or geographic habitability conditions. On the other hand, institutions have become more and more concerned about efficient use of their resources. Thus some transfer policies have become outmoded, but the basic requirements for transfers remain.

In order to ostensibly resolve this concern and in an effort to develop more efficient transfer policies, several rotation models have been developed within the past few years. Most of the models so developed can be categorized by two characteristics. One is the specific intent for use of the model; the other is the technique used to perform the analysis. Several models have been developed with the intent of mechanizing the assignment procedures of such a large personnel system, thereby hopefully reducing the workload of the personnel managers. Others have attempted to attack the

more basic problem of the development and evaluation of alternative transfer policies. The latter may be referred to as rotation planning models.

Several techniques have been used. In the models dealing with the assignment procedures, network flow algorithms have been used by Hatch [17,18] and mathematical programming algorithms have been used by Thorpe and Conner [35]. In the rotation planning models, several use simulation [27,28,37]. Others have used mathematical analytical approaches [8,34] and curve-fitting techniques [4].

In the development of the analytical planning model in chapter II of this thesis, the interdependence of the personnel inventory structure, called the billet structure throughout the thesis, the promotion structure and the rotation structure of the institution become quite apparent. Several models of personnel inventory structures have been developed. Among these, Markovian schemes have been used [1,29,30] as well as entity-simulation schemes [19,33]. More recently, Hayne [21] has developed an approach to Markovian schemes which allows two characteristics to be used to describe the states of the Markov model. Also, Marshall [26] has compared the use of a Markov model with the use of a cohort model for use in the prediction of the future personnel inventory of an institution. Grinold and Marshall [16] provide a thorough development of cross-sectional and longitudinal models, along with discussions about optimization and data formatting.

Only one model has been found that explicitly attempts to interrelate promotions, withdrawals, rotations and inventory. The model was developed by Hatch [20] and uses the entity-simulation technique with little documented analysis. The model in chapter II gives an analytical approach to the problem, and further develops an optimization scheme to evaluate different transfer policies.

The model presented in chapter III is concerned with the assignment process, so may be classed among those models. However, the emphasis is not on mechanizing the assignment process but rather on presenting different policies that the personnel managers might use to guide them in the process. Effort is made to relate the effect of the policies on the expenditures required to maintain the transfer process. These models provide planning aids which take into account factors previously ignored in the planning process. The model of chapter II is designed to be used on an interactive basis by the planner.

It is noted that the majority of the references listed above are to unpublished technical reports which were intended to focus on accounting problems and not on in-depth analysis of implications of policy changes. Although this thesis deals with the same or similar problems to those found in these reports, the emphasis here is on analytical approaches which give insight into the complex interactions among retention, promotion, rotation and inventory.

Other analytical work has been done in manpower planning and forecasting. Among early workers were Bartholomew [2],

Blumen, Kogen and McCarthy [3], Young and Almond [40] and Gani [11]. This work has focused on forecasting of inventory and labor turnover problems. More recently, Charnes and Cooper together with other authors including Niehaus, Sholtz, Stedry and Klingman (see list of references to [6]) have worked extensively in applications of mathematical programming to the mechanization of the personnel assignment process with emphasis on civilian manpower in the Department of the Navy.

B. CONTENTS AND SUMMARY

In this thesis, three analytical models which describe rotations in multi-location, multi-grade personnel systems are developed. Such systems are the basis of the personnel structure of many large institutions. The U.S. Navy is an example.

The three models are essentially independent, i.e. none depend on either of the other models for input or output. However, they are shown to interrelate macroscopically. That is, the insight provided by one may change the manner in which another is used.

In chapter II, some basic balance equations which relate the promotion structure, the rotation structure and the billet structure of a large institution are discussed. Two basic equations are developed in section B. These relations are used repeatedly throughout the remainder of the chapter. Section C contains the development of three different ways in which the billet structure may be characterized. These

prove useful later in the chapter. Sections D, E and F contain the development of the methodology required to demonstrate the effect of changes in the personnel system on the basic balance equations developed in section B. In section D, the necessary calculations to show the effect of various changes in the billet structure on the promotion structure holding other factors constant are performed. In section E, the effect of changes in the rotation structure on the promotion structure, holding the billet structure fixed, is described. Finally, in section F, changes which maintain a fixed promotion structure are described.

The next three sections of chapter II are devoted to the development of a methodology to compare the relative effectiveness of various changes in the system. Basically, from sections D, E and F we find that the same change in one component of the system can be compensated for by various changes in the other parts of the system. The choice of an "optimal" way of compensating for a specified change is then of interest. In section G, the results of the chapter up to that point are summarized, then the problem of comparative analyses of alternatives is introduced. A transportation problem format is introduced in section H as a method of effecting the comparison. Sensitivity analysis on the transportation problem is discussed in section I, both in general and for a specific case - that of a change in the billet structure for which the promotion structure is to be compensatory.

In section J some of the assumptions that were used earlier in the chapter are relaxed and the limitations that result with the relaxations are demonstrated. The chapter concludes with a discussion of possible extensions for the model.

In chapter III, attention is shifted from the flows of personnel within the institution to the problem of selecting a suitable replacement for an individual who is leaving some position within the institution. The model that is developed is called an Ehrenfest Decision Model because of some basic similarities to the Ehrenfest Urn model.

Following an introduction, in section B the Ehrenfest Decision Model is developed in the special case where the institution is confined to two separate locations. The model is developed under a simple decision rule, and then the decision rule is generalized. Approximations prove useful and are developed. In section C, the model is extended to institutions with three locations. Some comments are made about the difficulties to be expected if more than three locations are to be described.

The last section in the body of chapter III contains the development of diffusion approximations to the model of section B. Results from this approximation are useful in the description of transient cases. The conclusion to chapter III contains some remarks on the macroscopic influence of the results of chapter II on the interpretation of the results of the Ehrenfest Decision Model.

Our attention in chapter IV is on the problems encountered by the manager who is charged with making the selection of a suitable replacement. It is seen that a "good" choice of model parameters from chapter III may not be as good as was thought. The chapter begins by assuming in section B that the manager is operating with no restrictions on the number of personnel from which the selection is to be made and that he searches until the "right" replacement is found. The description is then extended to the case when the manager has a limited effort to be applied to finding a replacement, discussed in section C. In section D, restrictions on the number of personnel from which the replacement may be found are introduced. Restrictions are then made on both the number of personnel and the search effort, this being the basis for section E. In the conclusions to the chapter some possible extensions and also the influence of the results of chapter II on the results of this chapter are discussed.

II. RELATIONS BETWEEN PROMOTIONS, ROTATION RATES AND BILLET STRUCTURE IN A CLOSED INSTITUTION

A. GENERAL DISCUSSION

In this chapter we consider a manpower system for an institution whose personnel are in various locations and are further classified into ordered grades. An example of such an institutional manpower system is the enlisted force of the U.S. Navy where the locations could be the various ports and shore stations and the grades could be the nine pay grades within each skill category, or rating.

Personnel are rotated among the various locations; they are promoted between the various pay grades; requirements for a given skill category and pay grade can vary with both location and time period. We develop relations between promotions, rotation rates and the billet structure of such an institution. These relations are used to analyze the effects of changes in these factors under various assumptions.

Consider an institution in which personnel are classified into hierarchically ordered grades and in which personnel are hired only into the lowest grade. Again the enlisted force of a military institution serves as an example.

In this text, An;k denotes the kth assumption in chapter n; Dn;k denotes the kth definition in chapter n.

D2;1: A personnel system which hires only into the lowest grade is called a <u>closed system</u>.

The legacy of past recruiting, training and advancement severely constrains the range of current feasible decisions which can be used to control the personnel structure of an institution with a closed system. In addition to maintaining operation of the institution, the personnel managers must also ensure the development of present personnel resources to meet forecasted needs of future operations. Thus they assume responsibility for the training and advancement sequence to meet future requirements.

An institution using a closed system may wish to rotate personnel periodically. There may be several reasons for the rotations. For instance, one purpose for periodic rotations might be the desire to have the personnel exposed to a variety of job types. An ability to perform well in a variety of positions might be a measure used in the institution's promotion scheme. In this case, rotations might prove to be "career enhancing" for the individual.

For geographic, political, economic or other reasons, some of the locations where the institution uses personnel may be less desirable than other locations. The institution may desire to establish an equitable rotation plan influenced by such location factors. Specific positions may require extensive periods of family separation and this may be recognized by a policy of follow-on tours in positions more conducive to the maintenance of a close family life.

The institution may find that the rotation policy affects its retention. In a closed system the retention of

highly capable personnel is a significant factor in maintaining the caliber of the future leaders of the institution.

For instance, institutions involved in the extraction of raw mineral resources may conduct operations in areas deemed inadequate for continued habitation. Government employees are often utilized at "undesirable" locations. Without an effective rotation plan, both the mining and governmental institutions may have poor retention and hence possibly less effective future operations than would be possible with a more effective plan.

Whatever the reasons for establishing the rotation policy, the institution might choose to set minimum and maximum tour lengths for various positions at the various locations. Minimum tour lengths might be set to ensure that an individual's performance in a specific position can be properly evaluated. Maximum tour lengths might be prescribed in response to any combination of the above hypothesized reasons. Other factors peculiar to a particular institution might also be considered. In this thesis we assume that rotation planning is based on a set of given tour lengths. Let the tour length associated with a position at location i with grade k be denoted T(i;k).

D2;2: The <u>rotation rate</u> for a location i and a grade k, denoted R(i;k), is the reciprocal of the tour length for a position of grade k at location i, i.e., $R(i;k) = T(i;k)^{-1}$.

The billet structure of an institution is simply the number of positions of each grade at each location.

D2;3: The billets for a grade k at a location i, denoted b(i;k), give the number of personnel required in grade k at location i.

The personnel rotation scheme to be investigated is essentially based on close examination of the billet structure. As seen by D2;3, there are two attributes which each individual in the system is assumed to possess. The first attribute is location. The second attribute might be the pay grade of the individual, the skill group to which the individual belongs or another characteristic which is of interest to the rotation planners.

In the next section, we develop two basic relations which are used repeatedly throughout the remainder of the chapter. One arises from basic flow considerations, while the other comes from the assumption that the institution being investigated has a closed personnel system. The relations reflect the interactions between the billet structure of the system, the rotation structure of the system and the promotion scheme used in the system.

Section C contains the development of three different ways in which the billet structure may be characterized. These prove useful later in the thesis.

In sections D, E and F, we develop the methodology required to demonstrate the effect of changes in the system on the basic relations developed in section B. The next three sections, G, H and I, are devoted to the development of a methodology to compare the effectiveness of various changes to the system.

In section J, the effect of the relaxation of some of the assumptions in section B on the tractability of the problem is discussed. We then conclude the chapter with a discussion of some possible extensions to the work presented.

B. DEVELOPMENT OF THE BASIC BALANCE EQUATIONS

In this section, we develop two basic relations which are used repeatedly throughout the remainder of the chapter. One arises from basic flow considerations, while the other comes from the assumption that the institution being investigated has a closed personnel system.

It is assumed that the rotation rates R(i;k) are given. It is also assumed that the billet structure is known. By scanning the current list of employees and comparing each individual's date of assignment to his current position with the stated tour length, the rotational needs within the planning horizon can be determined, assuming that every vacated position must be filled. For simplicity the planning horizon is now taken to be one year.

D2;4: The rotational needs within planning period n, denoted r(i;k;n), are the <u>requirements</u> at location i for personnel who have second attribute k.

For ease of discussion in the following, the second attribute is taken to be pay grade. As stated in the previous section, the attribute used might also be skill group or other factor of interest to the planners.

D2;5: The number of personnel of grade k at location i whose tour is completed during period n (and hence must be rotated to another location) is denoted a(i;k;n) and called the <u>availabilities</u> at location i of grade k during period n.

Note that the locations are not necessarily physically separated. For instance, the U.S. Navy may designate one location as San Diego-Sea duty and another location San Diego-Shore duty since its rotational needs require some distinction between the two types of duty.

Since availabilities arise through completion of tours, there exists some transformation on past requirements which provides the current availabilities.

A2;1: There exists a square matrix Q which provides the transformation on past requirements giving current availabilities. Further, denoting by q(m,k) the element in the mth row and kth column of Q, we assume

$$a(i;k;n) = \sum_{m=1}^{G} r(i;m;n-T(i;m))q(m,k), \tag{2.1}$$

for all locations i, grades k and periods n of interest.

Note that q(m,k) gives the fraction of personnel who complete their tour in grade k, given they enter a location in grade m. Thus if there are G grades, Q is a G-by-G matrix.

The existence of Q postulated by A2;1 implies that a linear transformation on past requirements is sufficient to determine current availabilities. The possible lack of

agreement of this assumption with the structure of the system under investigation must be weighed against the tractability of the proposed model. The purpose of the current investigation is to aid in the development of insight into the system. As such, it is seen as a planning tool rather than a rigorous forecasting model. In this light, the tractability and resultant analytical results seem quite helpful.

It should be noted that the matrix Q has the structure of a finite state transient Markov chain. The significant difference between the Markovian model presented herein and Markov models used elsewhere lies in the interpretation of the matrix Q. In most cases, for example Bartholemew [2], Blumen *et al* [3], Charnes, Cooper and Niehaus [5], and Young [39], the Markov transition matrix is defined to provide the probability of transition between states during the course of a fixed time period. The period varies, sometimes being one year, sometimes one or more quarters of the year, etc, but the transition matrix is assumed to describe the flow during a period.

In this thesis, we assume the transition matrix to be defined over <u>tours</u>, and have established that tours vary over grades and locations. The resulting matrix of transition fractions (since we have defined fractional flows and not probabilistic transitions) is thus different from those of previous Markov models since there is no fixed period for which the transitions are measured.

Thus the elements of the matrix Q give the transformation on grades confounded by possible variances among tours. If

all tours within the institution are the same length, the matrix is interpreted in the same manner as previous models.

Suppose also that there exists a 1xG vector, w, where w(k) gives the fraction of the requirements for personnel of grade k which withdraw from the system prior to or during the period of their next availability.

A2;2: The withdrawal vector, w, is strictly positive.

Due to A2;2, the fraction of requirements for personnel of grade k which will be available upon completion of a tour will always be less than one. It is noted that A2;2 is more rigorous than required mathematically for the succeeding development. However, observation of the Navy personnel system indicates that A2;2 holds for that system. If a particular institution does not exhibit the withdrawal pattern of A2;2, then for all grades g from which there is no withdrawal one may set $w(g) = \varepsilon$ where ε is a small positive real number.

If each element of Q, q(m,k), is interpreted to give the fraction of personnel who entered their present tour (in response to a requirement) as grade m who will be available for reassignment as grade k upon completion of the tour, then the sum over k of q(m,k) will be less than one. In fact, $\sum_{k=1}^{G} q(m,k) + w(m) = 1$, for all grades $m = 1,\ldots,G$.

If the institution is neither gaining in population nor decreasing in population, every withdrawal must be matched by a recruit. Since the system is assumed to be closed, all recruits must enter in grade 1, the lowest grade. Then the

number of recruits during period n, denoted $r_o(n)$, is given by

$$r_o(n) = \sum_{i=1}^{L} \sum_{k=1}^{G} r(i;k;n-T(i;k))w(k). \qquad (2.2)$$

Equation (2.2) simply relates the recruiting requirements to the assumed withdrawal rate over all locations i, $i = 1,\ldots,L$ and grades k, $k = 1,\ldots,G$ and to historical requirements.

Let us assume that the requirements at n-T(i;k) are equal to the requirements in period n for all locations i and grades k. This is a "weak" form of stationarity in that some cyclical variations may occur so long as the requirements repeat every T(i;k) periods.

Let a(i;n) be a 1xG vector giving the availabilities at location i during period n, where the elements of a(i;n) are a(i;k;n). Similarly let r(i;n) be a 1xG vector giving the requirements at location i. Then equation (2.1) may be written for any period n as

$$a(i;n) = r(i;n)Q. \qquad (2.1')$$

If all those available to move must move due to the considerations of the preceeding section and if all requirements of the system must be met, then

$$\sum_{i=1}^{L} a(i;n) = \sum_{j=1}^{L} r(j;n) - (r_o(n),0,\ldots,0),$$

where the last vector on the right-hand-side gives the increase in availabilities of personnel of grade 1 due to recruitments.

Substituting relation (2.1') into the left-hand-side of the above equation and rearranging terms, we obtain

$$(I-Q) \sum_{i=1}^{L} r(i;n) = (r_o(n),0,\ldots,0),$$

or, if $(I-Q)^{-1}$ exists where I is the identity matrix of appropriate dimension,

$$\sum_{i=1}^{L} r(i;n) = r_o(n)N_1, \qquad (2.3)$$

where N_1 is the first row of $(I-Q)^{-1}$. The notation $N = (I-Q)^{-1}$ follows Kemeny and Snell, [23], who call N the <u>fundamental matrix</u>. Since Q may be interpreted as a substochastic matrix due to A2;2, the fact that N exists is well known. Kemeny and Snell ([23], p. 46, Theorem 3.2.4) give the interpretation of the (i,j)th element of N as the expected number of visits to state j by an arbitrary entry into state i. Each element of the vector equation (2.3) may be stated $\sum_{i=1}^{L} r(i;k;n) = r_o(n)N_1(k)$. Thus, element by element (2.3) equates the total requirements for personnel of grade k during period n to the product of the number of entries into the system during period n and the expected number of visits to grade k by an arbitrary entry into the system.

A disadvantage of the above model formulation is that the factors under the control of management personnel such as billet structure and rotation rates are not explicit in the structure of the relations (2.2) and (2.3). Thus, the effect of varying one or both of these factors is not obvious.

We now introduce the planning factors explicitly into relations (2.2) and (2.3). Due to the assumed cyclical nature of the requirements, i.e. the assumption that $r(i;k;n) = r(i;k;n-T(i;k))$ for all locations i and grades k, every one of the $b(i;k)$ billets for personnel of grade k at location i must be replaced every $T(i;k)$ periods, and during this time each will be replaced only once. Hence

$$b(i;k) = \sum_{n=m}^{T(i;k)+m-1} r(i;k;n),$$

where the period in which the counting starts, m, is arbitrary. If the system is stationary in the usual sense, i.e., $r(i;k;n)$ is independent of the period n, the above summation gives $b(i;k) = r(i;k;n)T(i;k)$.

A2;3: The requirements $r(i;k;n)$ are independent of the period n.

We now suppress the n in the previous notation, setting $r(i;k;n) = r(i;k)$ for all locations i and grades k. Now requirements $r(i;k)$ are given by $b(i;k)/T(i;k)$ or since $R(i;k) = T(i;k)^{-1}$, $r(i;k) = b(i;k)R(i;k)$. Substituting into (2.2) and (2.3) respectively, we have

$$r_o = \sum_{i=1}^{L} \sum_{k=1}^{G} b(i;k)R(i;k)w(k) \qquad (2.4)$$

and

$$r_o N_1(k) = \sum_{i=1}^{L} b(i;k)R(i;k), \quad k = 1,\ldots,G.$$

The last equation may be stated more succinctly as

$$r_o N_1 = \underline{1} B \odot R \tag{2.5}$$

where the symbol $\odot$ means element by element multiplication of the matrices B and R and $\underline{1}$ is the sum vector of the appropriate length. Equations (2.4) and (2.5) are called the "basic balance equations" for the system.

Much notation has been introduced in this section that is used extensively in the sections which follow. For this reason, a summary of notation is included in Table 2.1.

The next section contains a discussion of some alternative ways in which the billet structure may be characterized. These are of use in subsequent examination of the implications of the basic balance equations, (2.4) and (2.5).

C. FRACTIONAL BILLET STRUCTURES

This section develops three different ways in which the billet structure may be characterized. The characterizations are not independent, but each is useful in specific situations later in the thesis.

Let b be the total number of personnel in the system i.e. $b = \sum_{i=1}^{L} \sum_{k=1}^{G} b(i;k)$. Then the (i,k)th element of $b^{-1}B$ gives the fraction of personnel which are of grade k at location i. Denote this fractional distribution matrix by Γ, with (i,k)th element $\gamma(i;k) = b(i;k)/b$. One sees immediately that the fractional matrix can be substituted for the billet structure matrix B in (2.4) and (2.5), and that these equations are linear in the total system size b. Hence

Table 2.1.

SUMMARY OF NOTATION

- L is the number of locations within the system, indexed i.
- G is the number of grades (or other second attribute) within the system, indexed k.
- $T(i;k)$ denotes the Tour length in integral periods for a person of grade k at location i.
- $R(i;k)$ denotes the rotation rate for a position of grade k at location i and equals $1/T(i;k)$.
- R is a LxG matrix with elements $R(i;k)$.
- $b(i;k)$ denotes the number of billets of grade k at location i.
- B is a LxG matrix with elements $b(i;k)$.
- $r(i;k;n)$ denotes the number of personnel of grade k required to be assigned to location i during period n. In the stationary system, the n is suppressed.
- $r(i;n)$ or $r(i)$, a 1xG vector with elements $r(i;k;n)$ or $r(i;k)$, respectively.
- $r(n)$ or r, a 1xGL vector defined by $r = (r(1),...,r(L))$.
- $a(i;k;n)$ denotes the number of personnel of grade k which are available to move from location i during period n. In the stationary system, the n is suppressed. This is an element of $a(i;n)$, $a(i)$, $a(n)$ and a in the same manner as $r(i;k;n)$ is an element of the corresponding requirement vectors.
- Q is a GxG matrix giving the fractional flows among grades during a tour.
- w is a 1xG vector giving the fraction of personnel who enter a tour in grade k who choose to leave the system upon completion of the tour. Elements referred to as <u>withdrawal</u> rates.
- r_o (or $r_o(n)$), the number of recruits (during period n).
- N_1, a 1xG vector which is the first row of $(I-Q)^{-1}$.

policy alternatives may be examined using the fractional distribution matrix and results applied to an institution of any size. It is noted that Γ has dimension LxG.

Let $b(+;k) = \sum_{i=1}^{L} b(i;k)$, the total number of personnel of grade k in the system. Denote by Φ a GxL matrix with (k,i)th element $\phi(k;i) = b(i;k)/b(+;k)$. That is, the (k,i)th element of Φ gives the fraction of personnel of grade k who are located at i. Note that

$$\gamma(i;k) = \phi(k;i)b(+;k)/b. \tag{2.6}$$

Similarly, let $b(i;+) = \sum_{k=1}^{G} b(i;k)$. Thus $b(i;+)$ gives the population at location i. Denote by Θ a LxG matrix with elements $\theta(i;k) = b(i;k)/b(i;+)$. Here the (i,k)th element of Θ gives the fraction of personnel at location i who are of grade k. Note here that

$$\gamma(i;k) = \theta(i;k)b(i,+)/b. \tag{2.7}$$

From (2.6) and (2.7), we see immediately that

$$b(i;k) = b(+;k)\phi(k;i) \tag{2.8a}$$

$$= b(i;+)\theta(i;k), \tag{2.8b}$$

and that the row sums of Φ and Θ are unity. Summing (2.8a) over k and (2.8b) over i gives

$$b(i;+) = \sum_{k=1}^{G} b(+;k)\phi(k;i) \tag{2.9a}$$

and

$$b(+;k) = \sum_{i=1}^{L} b(i;+)\theta(i;k). \tag{2.9b}$$

Define $b_{L\cdot}$ as a 1xL vector with elements b(i;+) and $b_{\cdot G}$ as a 1xG vector with elements b(+;k). Then the preceeding relationships may be expressed in vector-matrix notation as $b_{L\cdot} = b_{\cdot G}\Phi$ and $b_{\cdot G} = b_{L\cdot}\Theta$. Substituting one into the other gives us the relations

$$b_{L\cdot} = b_{L\cdot}\Theta\Phi \tag{2.10a}$$

and

$$b_{\cdot G} = b_{\cdot G}\Phi\Theta. \tag{2.10b}$$

Lemma 2.1. The matrices $\Theta\Phi$ and $\Phi\Theta$ are non-negative and have row sums equal unity.

Proof: Non-negativity follows from the definition of Θ and Φ. First consider $\Theta\Phi$. Then

$$\sum_{i=1}^{L} (\Theta\Phi)_{ji} = \sum_{i=1}^{L} \sum_{m=1}^{G} \theta(j,m)\phi(m,i)$$

$$= \sum_{m=1}^{G} \theta(j,m) \sum_{i=1}^{L} \phi(m,i)$$

$$= \sum_{m=1}^{G} \theta(j,m) = 1.$$

The result for $\Phi\Theta$ follows by a symmetrical argument.

By virtue of the above lemma, both $\Theta\Phi$ and $\Phi\Theta$ have the properties of a Markov matrix, which we assume will be regular. Therefore, using the well known steady state distributional result of Markov theory (for example, see Kemeny and Snell [23]) equations (2.10a) and (2.10b) have solutions $b_{L\cdot}$ and $b_{\cdot G}$ respectively which are unique up to a multiplicative constant. Hence given the matrices Φ and Θ and the

total system size b, the marginal billet structures by location and grade are uniquecly determined.

In summary we have characterized the billet structure in three forms, Γ, Φ, and Θ. These are used in subsequent sections in the analysis of system changes brought about by changes in the billet structure or rotation rates.

D. ABSORBING BILLET CHANGES THROUGH CHANGES IN THE MATRIX Q

1. Changes in a Single Element of B

Suppose the population of personnel of a specific grade at a specific location is increased or decreased. Suppose also that there are no other changes in billets throughout the system. Thus the total population of the institution changes by the same amount. In order to maintain the basic balance equations ((2.4) and (2.5)) there have to be changes in some other factors. In this section we fix tour lengths and withdrawal rates and observe the effect of the billet change on the matrix Q. If we assume the second attribute which describes the population is pay grade, then Q is a description of the promotion scheme.

Suppose we wish to change the number of positions at location ℓ in grade g, i.e. the quantity $b(\ell;g)$. We wish to express r_o and the vector N_1 as functions of the change in $b(\ell;g)$. To do this, we parameterize the population at location ℓ of grade g by denoting the population following a change as $b'(\ell;g)$ and setting $b'(\ell;g)=(1+\alpha)b(\ell;g)$ Here α is the fractional change in $b(\ell;g)$, the control variable. It is straightforward to show that r_o is linear in

α. From (2.4) we obtain

$$r_o(\alpha) = r_o + w(g)R(\ell;g)b(\ell;g)\alpha \tag{2.11}$$

where

$$r_o = \sum_{i=1}^{L} \sum_{k=1}^{G} b(i;k)R(i;k)w(k).$$

Also, from (2.5) and (2.11), it is trivial to show

$$N_1(k;\alpha) = \frac{\rho(k)}{r_o + w(g)R(\ell;g)b(\ell;g)\alpha}, \quad k \neq g \tag{2.12a}$$

where

$$\rho(k) = \sum_{i=1}^{L} b(i;k)R(i;k),$$

or $\rho(k) = r_o N_1(k)$ from equation (2.5). We also obtain when $k=g$

$$N_1(g;\alpha) = \frac{\rho(g) + R(\ell;g)b(\ell;g)\alpha}{r_o + R(\ell;g)b(\ell;g)w(g)\alpha}. \tag{2.12b}$$

Recall that N_1 is the first row of $(I-Q)^{-1}$. Thus the above equations do not show explicitly how Q changes with α. In order to proceed we assume the following special structure of Q.

A2;4: The promotion scheme used by the institution allows no demotions; no individual is allowed to advance more than one grade during any given tour.

Recall that the element $q(m,k)$ of Q is the fraction of personnel who complete their tour in grade k, given they entered their present tour in grade m. For the

special structure given by A2;4 let $q(m,m) = q(m)$ and $q(m,m+1) = p(m)$. All other $q(m,k)$ are taken to be zero.

Hence Q is seen to have the following structure:

$$Q = \begin{pmatrix} q(1) & p(1) & 0 & 0 & 0 & 0 & & \\ 0 & q(2) & p(2) & 0 & 0 & 0 & & \\ 0 & 0 & q(3) & p(3) & 0 & 0 & & \\ & & & & \vdots & & & \\ & & & 0 & 0 & 0 & q(G-1) & p(G-1) \\ & & & 0 & 0 & 0 & 0 & q(G)' \end{pmatrix}.$$

In this special case, let $n(i) = (1-q(i))^{-1}$. Then N_1 can be shown to have the following structure:

$$N_1 = (n(1), n(1)p(1)n(2), \ldots, n(G)\prod_{k=1}^{G-1} n(k)p(k)). \quad (2.13)$$

With this form and recalling that $q(i) + p(i) + w(i) = 1$, we can iteratively calculate the elements of Q from N_1 for any fixed w. We note that with the special structure imposed by assumption A2;4, the specification of the billet structure B, the rotation rates R and the withdrawal rates w, is sufficient to determine a unique promotion matrix Q.

We now give an example, the basic elements of which are used throughout the rest of the chapter.

Example 2.1. Suppose we have an institution with four locations and five pay grades. Suppose the billet structure B, the rotation rates R and the withdrawal rates w are given by:

$$B = \begin{pmatrix} 300 & 240 & 180 & 70 & 35 \\ 600 & 455 & 230 & 150 & 75 \\ 300 & 240 & 180 & 120 & 60 \\ 1140 & 600 & 440 & 280 & 80 \end{pmatrix},$$

$$R = \begin{pmatrix} .333 & .333 & .333 & .5 & .5 \\ .333 & .333 & .5 & .5 & .5 \\ 1 & 1 & 1 & 1 & 1 \\ .333 & .5 & .5 & .5 & 1 \end{pmatrix},$$

and $w = (\ .1\ .3\ .2\ .3\ .4)$.

Then the promotion matrix Q required to maintain the given billets and tour pattern is

$$Q = \begin{pmatrix} .354 & .546 & & & \\ & .306 & .394 & & \\ & & .471 & .329 & \\ & & & .489 & .211 \\ & & & & .600 \end{pmatrix}.$$

Example 2.2. Suppose now that we wish to change the billets at location 4 in grade 1, i.e. $b(4;1)$, to 800 from the current 1140. Maintaining R and w fixed, we ask for the changes in Q required by this change in billet structure. The new Q is

$$Q = \begin{pmatrix} .282 & .618 & & & \\ & .306 & .394 & & \\ & & .471 & .329 & \\ & & & .489 & .211 \\ & & & & .600 \end{pmatrix}.$$

Note that the decrease in billets of grade 1 has resulted in an increase in promotion rate from grade 1 to grade 2. This is to be expected, since a smaller community, i.e. the new population of grade 1, is being required to support the same population of grade 2 as was required previously. Hence a larger proportion of grade 1 will be promoted during each tour. Perhaps the surprising result from example 2 is that <u>only</u> the elements q(1) and p(1) changed with the change in b(4;1). In fact, it is true in general that for a change in billets of grade k, only those elements of Q, q(i) and p(i), for i less than or equal to k, will change. We now show this result analytically, but first we need the following lemma.

<u>Lemma 2.2</u>. Assumption A2;4 and equation (2.13) imply that the element q(k) of Q may be expressed

$$q(k) = 1 - \frac{1}{N_1(k)} \{ 1 - \sum_{j=1}^{k-1} w(j)N_1(j) \}. \tag{2.14}$$

Proof. From (2.13) we have

$$N_1(1) = n(1) = 1/(1-q(1)),$$

from which

$$q(1) = 1 - N_1(1)^{-1}.$$

Assume now that (2.14) holds for k. We show that this implies that (2.14) holds for k+1. First we observe that (2.13) implies that

$$N_1(k+1) = \frac{N_1(k)p(k)}{1 - q(k+1)} .$$

From the above, we obtain

$$q(k+1) = 1 - \frac{N_1'(k)p(k)}{N_1(k+1)} .$$

But we also have that $p(k) = 1 - q(k) - w(k)$. Substituting for $p(k)$ in the preceeding and cancelling terms, we obtain the result. Hence the lemma is shown by induction.

Theorem 2.3. For a fractional change α in billets of grade g at location ℓ, the elements of Q, under A2;4, are given by

$$q(k;\alpha) = \begin{cases} \dfrac{1}{\rho(k)} \{C_1(k) - w(g)R(\ell;g)b(\ell;g)\alpha\}, & k=1,\dots,g-1 \\ 1 - w(g) - \dfrac{r_o - \sum_{j=1}^{g} w(j)\rho(j)}{\rho(g) + R(\ell;g)b(\ell;g)\alpha}, & k=g \\ C_1(k), & k=g+1,\dots,G; \end{cases}$$

where

$$C_1(k) = \rho(k) - r_o + \sum_{j=1}^{k-1} w(j)\rho(j).$$

Proof: The proof follows by straightforward substitution of equations (2.12) into (2.14).

Example 2.3. Suppose now that we wish to change the billet structure of example 1 by setting $b(1;5)$, the billets in location 1 of grade 5, to 1000. Again fixing R and w, we ask for the promotion matrix required to maintain the specified structure. We obtain

$$Q = \begin{pmatrix} .157 & .743 & & & \\ & .056 & .644 & & \\ & & .136 & .664 & \\ & & & -.032 & .732 \\ & & & & .600 \end{pmatrix} .$$

Example 2.3 illustrates that there are limitations on the ability of the system to absorb changes in billets only through changes in the promotion scheme used. The matrix Q must be non-negative; hence the element q(4,4) indicates that the promotion scheme will not support the desired change in billet structure.

Next we develop relations which provide for a change in a specific billet to be absorbed throughout the institution in such a manner as to preserve both the total population and the population within each grade.

2. Change in One Element of B Absorbed Proportionately Throughout the System

Suppose now that the increase or decrease in the population $b(\ell;g)$ is restricted so that neither the population in grade g, $b(+;g)$, nor the system population b is allowed to change. Thus the location populations $b(i;+)$ must change. There are two cases that would be of general interest in this instance.

In the first case, a sub-unit of location ℓ may move as a unit to location k, $k \neq \ell$. This would be the case, for instance, if a firm engaged in heavy construction completed a project in Idaho and simultaneously acquired a contract in Arizona. This case could be easily handled by two iterations of the scheme in the above sub-section.

In the second case, the firm might complete operations in one area and not immediately desire to move everyone to a specific new area of operations. In this case, in order to retain valuable expertise the firm may redistribute the excess personnel throughout the system. We examine this case, assuming the distribution is to be made so that each location i gains in population in ratio to its population of grade g. Now let $b'(\ell;g)$ denote the population $b(\ell;g)$ following the change, where $b'(\ell;g) = (1+\alpha)b(\ell;g) = b(+;g)(1+\alpha)\phi(g;\ell)$. We know that the sum over i of $\phi(g;i)$ is one. Also, letting $\phi'(g;\ell) = (1+\alpha)\phi(g;\ell)$ and $\phi'(g;i) = (1+\beta)\phi(g;i)$ for $i\neq\ell$ to ensure proportional distribution, we must have that the sum over i of $\phi'(g;i)$ is one. We note that the prime notation implies the value following the change. The accumulated changes must sum to zero. Hence

$$\beta \sum_{i\neq\ell} \phi(g;i) + \alpha\phi(g;\ell) = 0.$$

Recognizing that $\sum_{i\neq\ell} \phi(g;i) = 1 - \phi(g;\ell)$, we obtain

$$\beta = - \alpha\phi(g;\ell)/(1-\phi(g;\ell)).$$

We now have that the new billet size for personnel of grade g at locations i, $i\neq\ell$, is given by $b'(i;g) = b(+;g)(1+\beta)\phi(g;i) = (1+\beta)b(i;g)$. We are now able to develop the expressions for the total recruits r_o and the vector N_1, and thus Q, from the basic balance equations. By straightforward substitution and some algebra, we obtain

$$r_o(\alpha) = r_o + K_1 w(g)\alpha, \tag{2.15}$$

where

$$K_1 = b(\ell;g)R(\ell;g) - \frac{\phi(g;\ell)}{1 - \phi(g;\ell)} \sum_{\substack{i=1 \\ i\neq\ell}}^{L} b(i;g)R(i;g).$$

We also obtain, for $k\neq g$,

$$N_1(k;\alpha) = \frac{\rho(k)}{r_o + K_1 w(g)\alpha} \tag{2.16a}$$

and

$$N_1(g;\alpha) = \frac{\rho(g) + K_1\alpha}{r_o + K_1 w(g)\alpha}, \tag{2.16b}$$

where ρ is given in (2.12).

We note that (2.15) and (2.16) correspond very closely to equations (2.11) and (2.12). In fact, the only difference is that in the relations developed in this section, the constant K_1 takes the place of the constant product $b(\ell;g)R(\ell;g)$ of the previous work. Hence, the results of theorem 2.3 extend to this case with the proper change of constants.

We also note that K_1 may be written

$$K_1 = \frac{\phi(g;\ell)}{1 - \phi(g;\ell)} \left(b(+;g)R(\ell;g) - r_o(0)N_1(g;0)\right).$$

The interpretation of this equation must await future discussion, but we can note that should the term in parentheses be zero, then equations (2.15) and (2.16) would be constant in α.

With assumption A2;4 we can now obtain the new promotion matrix Q using (2.13) and (2.14). We conclude this sub-section with an example. In the next sub-section we

extend our results to the case where the entire force at some location is scaled up or down.

Example 2.4. Suppose now we start with the structure given in example 1, and we wish to change b(4;1) to 800 as was done in example 2. Suppose, however, that now we must maintain b(+;1) and b. Under this restriction and following the results of this section, we obtain the new billet structure B as

$$B = \begin{pmatrix} 385 & 240 & 180 & 70 & 35 \\ 770 & 455 & 240 & 150 & 75 \\ 385 & 240 & 180 & 120 & 60 \\ 800 & 600 & 440 & 280 & 80 \end{pmatrix} .$$

Note that the first column of B still sums to 2340, the sum of the first column of B in example 2.1. Thus the drop in billets in b(4;1) has been distributed among the other locations.

The new promotion matrix is

$$Q = \begin{pmatrix} .382 & .517 & & & \\ & .306 & .394 & & \\ & & .471 & .329 & \\ & & & .489 & .211 \\ & & & & .600 \end{pmatrix} .$$

Note again that only elements q(1) and p(1) change. Also note that while in example 2.2 p(1) increased from .546 to .618, here p(1) has decreased to .517. The reason for this decrease is more subtle than was the reason for the increase in p(1) in example 2.2. In the definition of p(1) as an

element of Q, it is seen that this number gives the fraction of personnel who enter a tour in grade 1 who subsequently complete the tour in grade 2. In the restructuring of the billet matrix B, we have increased b(3;1) from 300 to 385. Looking at R, we note that T(3;1) = 1 while the other locations have tour lengths of 3 years for personnel of grade 1. Hence under the restructuring as given, more personnel complete a tour each year. Hence the "average" number of promotions from grade 1 to grade 2 during a tour must decrease in order to maintain the same personnel flow into grade 2.

In the next section we generalize the results of this section to the case were all grades at a specified location are scaled, reflecting a general increase or decrease in force size at the specified location.

3. Scaling of Billets of all Grades at a Specified Location

Suppose now that we wish to scale (up or down) the population at a specified location, denoted ℓ. In this case we retain the proportional structure of the institution, but change the population at location ℓ to reflect a general build-up or decay in operations at the location.

Here we write $b'(\ell;+) = (1+\alpha)b(\ell;+)$. From this we obtain $b'(\ell;g) = b'(\ell;+)\theta(\ell;g) = (1+\alpha)b(\ell;+)\theta(\ell;g) = (1+\alpha)b(\ell;g)$ for all grades g. We also obtain $b = b + \alpha b(\ell;+)$. From the basic balance equations we obtain

$$r_o(\alpha) = r_o + K_2\alpha,$$

where

$$K_2 = \sum_{k=1}^{G} b(\ell;k)R(\ell;k)w(k).$$

We also obtain

$$N_1(k;\alpha) = \frac{\rho(k) + b(\ell;k)R(\ell;k)\alpha}{r_o + K_2\alpha}.$$

From (2.14) we obtain

$$q(k;\alpha) = \frac{C_1(k) + K_2'(k)\alpha}{\rho(k) + b(\ell;k)R(\ell;k)\alpha},$$

where

$$K_2'(k) = b(\ell;g)R(\ell;g) - \sum_{j=k}^{G} b(\ell;j)R(\ell;j)w(j).$$

We thus observe that the result is more complicated in nature than previous results. Note that $q(k;\alpha)$ is not linear in α for any grade. Also note that $\partial q(k;\alpha)/\partial\alpha$ is not zero for any particular set of grades as it was in the previous work. This is expected, since now we have changed all grades at some location instead of a single grade as was done previously. We conclude this section with an example.

Example 2.5. Suppose we begin with a system such as that in example 2.1. Suppose now that we wish to scale down location 4 by a factor which will bring $b(4;1)$ to 800, the figure used in example 2.2. Suppose also that the effect of the change is to be only on the promotion scheme, leaving the rotation rates and withdrawal rates constant. The billet structure becomes

$$B = \begin{pmatrix} 300 & 240 & 180 & 70 & 35 \\ 600 & 455 & 230 & 150 & 75 \\ 300 & 240 & 180 & 120 & 60 \\ 800 & 421 & 309 & 196 & 56 \end{pmatrix} .$$

The resulting promotion matrix Q is

$$Q = \begin{pmatrix} .354 & .546 & & & \\ & .306 & .394 & & \\ & & .472 & .328 & \\ & & & .491 & .209 \\ & & & & .600 \end{pmatrix} ,$$

which we note is very close to the Q used in example 2.1. This may be explained by the fact that an across-the-board cut was made in personnel in the system. Due to the fact that all grades were affected in a similar fashion, there was not a significant change in the proportions of the grade structure.

E. ABSORBING CHANGES IN ROTATION RATES THROUGH CHANGES IN THE MATRIX Q

Suppose the rotation rate for a specific grade at a specific location is to be changed. Suppose also that there is to be no other change in rotation rates throughout the system. As was noted in sub-section D.1 above, in order to maintain the basic balance equations there will have to be changes in other factors. In this section we fix the billet structure and the withdrawal rates and observe the effects on the matrix Q.

Examination of the basic balance equations (2.4) and (2.5) reveals that the effects on the matrix Q of changes

in R with B fixed are symmetrical. Suppose we are interested in the effects of a change in rotation rate for positions of grade g at location ℓ, i.e. the quantity $R(\ell;g)$. Denoting the value of $R(\ell;g)$ after the change by $R'(\ell;g) = (1+\alpha)R(\ell;g)$, we note that after substitution into (2.4) and (2.5), the result shows the same mathematical relations as were developed in (2.11) and (2.12).

Also in the same manner as was done above, simplifying assumption A2;4 can be made regarding the structure of the matrix Q, from which the changes in Q can be directly calculated. Since there is nothing different about these calculations once the vector N_1 is known, we refer back to section D.1 for the method and complete this section with an example.

Example 2.6. Suppose that we wish to change the rotation rate R(4;1) to .234. We desire to leave the billet structure and withdrawal rates constant, absorbing the change in the promotion system. The resultant Q is that of example 2.2, and since the product b(4;1)R(4;1) is the same in the two examples, this result confirms the similarity in the mathematical structure for the two situations.

F. CHANGES WHICH PRESERVE THE MATRIX Q

1. Change in a Single Element of B

Suppose the number of billets of grade g at location ℓ is to be changed. It is desired that the change not affect the "promotion" matrix Q. Since fixing Q implies that w and N_1 are fixed, the change must be balanced by a change in the

rotation structure R. Suppose only $R(\ell;g)$ is to be allowed to change with the change in $b(\ell;g)$.

With the above restrictions, we denote the change in $b(\ell;g)R(\ell;g)$ by $b'(\ell;g)R'(\ell;g) = (1+\alpha)b(\ell;g)R(\ell;g)$, obtaining functional expressions for r_o and N_1 as

$$r_o(\alpha) = r_o + w(g)b(\ell;g)R(\ell;g) \; ,$$

$$N_1(k;\alpha) = \rho(k) \; \{r_o + w(g)b(\ell;g)R(\ell;g)\alpha\}^{-1},$$

for $k \neq g$, and

$$N_1(g;\alpha) = \frac{\rho(g) + b(\ell;g)R(\ell;g)\alpha}{r_o + b(\ell;g)R(\ell;g)w(g)\alpha} \; .$$

From the restrictions, $N_1(k;\alpha)$ must be constant. Therefore

$$\alpha = \frac{\rho(k) - N_1(k;\alpha)r_o}{w(g)R(\ell;g)b(\ell;g)}$$

$$= \frac{r_oN_1(k) - r_oN_1(k;\alpha)}{w(g)R(\ell;g)b(\ell;g)}$$

$$= 0.$$

Hence $b(\ell;g)R(\ell;g)$ must be constant. Hence we are assured that the basic balance equations hold.

Thus under the restriction of a fixed Q, as billets are increased, the rotation rate must decrease which means that the tour length is increased. Here one can see the direct relation between the number of billets that can be supported at a given location and the rotation rates or tour lengths. Conversely the above relations can easily be developed for changes in "supportable" billets with changes in rotation rates.

Example 2.7. Starting with the structure of example 2.1., suppose we wish to change b(4;1) to 800 as was done in example 2.2, but this time we wish to compensate for the change by modifying the rotation rate R(4;1). Following the results of this section, it is readily determined that R'(4;1) is .475 for a new tour length of 2.105 years.

2. <u>Change in Rotation Rate R(ℓ;g) to be Absorbed Throughout Grade g</u>

As we generalize from a single element change in R, discussed in section E, to a compensatory change throughout the system, the analogy of the rotation rate matrix, or rotation structure, to the billet structure fails to hold. Although the mathematical structure carries through, the intuitive meaning of establishing rates which in some sense compensate for a specified change in the rate at some location for one grade is obscured. We specify the sense intended and develop relations which preserve the basic balance equations.

Consider the effect on relation (2.5) if the rotation rate is independent of location, but remains a function of grade. Let R(*;k) represent this special type of rotation scheme, k = 1,...,G. Then (2.5) becomes

$$R(*;k)\sum_{i=1}^{L} b(i;k) = r_o N_1(k).$$

But the summation on the left-hand-side is b(+;k). Writing out the right-hand-side of the above equation, we obtain

$$R(*;k)b(+;k) = \sum_{i=1}^{L} b(i;k)R(i;k)$$

or, $$R(*;k) = \sum_{i=1}^{L} b(i;k)b(+;k)^{-1}R(i;k),$$

and $R(*;k)$ is seen to be a weighted average of the rotation rates. Further, using (2.8a) we obtain

$$R(*;k) = \sum_{i=1}^{L} \phi(k;i)R(i;k).$$

We are now able to interpret the constant K_1 in equations (2.15) and (2.16). We see that for all locations ℓ and grades g such that $R(\ell;g) = R(*;g)$, the billet structure can be changed without affecting either $r_o(\alpha)$ or the vector $N_1(\alpha)$.

Suppose we start with a rotation scheme described by $R(*;k)$, $k = 1,\ldots,G$. Suppose now that the rotation rate $R(\ell;g)$ for a location ℓ and grade g is to be varied from $R(*;g)$, independent of other considerations. We now develop the calculations which give a new weighted average rotation rate for positions of grade g at locations other than ℓ. This is done while at the same time ensuring that (2.5) holds. Denote the new rate $R'(*;g)$.

Since we desire that the resultant weighted average have the property that

$$R'(*;g)\sum_{i\neq\ell} b(i;g) + R(\ell;g)b(\ell;g)$$

be constant, we fix $R(*;g)$ as the desired constant term. Now $R'(*;g)$ can be written as a function of $R(\ell;g)$, giving

$$R'(*;g;R(\ell;g)) = C_1 + K_3R(\ell;g),$$

where

$$C_1 = R(*;g) / b(+;g)(1 - \phi(g;\ell))$$

and

$$K_3 = \phi(g;\ell) / (1 - \phi(g;\ell)).$$

It is essential at this point to check relation (2.4) to ensure that r_o has not changed. Note that the above development is concerned with the changes in a specific grade, denoted g. If r_o changes due to a change in $R(\ell;g)$ and $R(*;g)$, then this change will affect all relations (2.5) and not just the one in the variable g. Checking is a matter of straightforward substitution and it may readily be verified that r_o does not change with the above scheme. Also since the calculations ensure that the product $r_oN_1(g)$ remains constant and r_o is constant, then $N_1(g)$ is constant. Since none of the other relations (2.5) are involved, then the vector N_1 is constant with the prescribed changes in rotation rates. Hence the "promotion" matrix Q is constant.

Thus we have shown that rotation rates may be changed so as to not affect other parameters of the model. One word of caution is due, however. As specific locations or specific location/grade elements are selected for the "special" consideration regarding their rotation rates, the weighted average rotation rate assigned the remaining positions may become unreasonable small (or large), implying unreasonable long (or short) tours.

An APL program was written which assumes the billet structure B and promotion matrix Q are given. The program calculates rotation rates based on the weighted average scheme described in this section. The program is interactive in the sense that the user is asked to select any locations to be given "special" consideration. Changes that result from the calculations are accomplished so as to not change the matrix Q or billet structure B. The program is given in an appendix.

Example 2.8. Suppose now that we wish to change R(4;1) to .234, as we did in example 2.6. Suppose further that we wish to absorb the change by changing other rotation rates, holding other factors fixed. We obtain the new rotation rate matrix

$$R = \begin{pmatrix} .396 & .333 & .333 & .5 & .5 \\ .396 & .333 & .5 & .5 & .5 \\ 1.189 & 1 & 1 & 1 & 1 \\ .234 & .5 & .5 & .5 & 1 \end{pmatrix}$$

which is observed to change only in the first column, i.e. the effect of the change is confined to the grade being changed. Calculation of Q using this R and the B and w of example 2.1 will confirm that Q remains the same under both rotation schemes. Thus the effect is felt only in R.

G. SUMMARY AND INTRODUCTION TO COMPARATIVE ANALYSIS OF ALTERNATIVES

We now summarize the different approaches to rotation planning that have been discussed and generalize the results of the preceeding sections. The special cases used thus far will motivate the ensuing discussion.

As we re-examine the special cases already discussed, a significant fact is evident. In the structure where two attributes are jointly used in manpower planning, many restrictions were required in order to obtain analytical results. The underlying reason for most of these restrictions was the otherwise ensuing lack of uniqueness to the systems of equations generated.

Whatever restrictions are placed on the model and whatever scenario is thus selected for observation by the personnel/manpower managers, there comes a time when one must ask which of several proposed alternatives is in some sense "best." In the usual case, there seems to be a hierarchy of factors considered in selecting the best of a given set of alternatives. For instance, the output of someone else's planning model has probably dictated the billet structure for the manpower planner. The most that the manpower planner can hope to accomplish with respect to altering the billet structure is to point out significant problems associated with the imposed billets. For instance, there may be some hidden costs associated with one location. By pointing out these additional costs to his supervisors, the personnel manager may show that the operation in question is not as

profitable as the higher echelon planners thought. This approach may not always be necessary. Within his own purview there may be many alternative ways in which operations may be conducted. It is the manpower planner's job to select an acceptable alternative or set of alternatives.

The sections that follow focus on the transfer cost incurred due to a particular billet structure, rotation structure and promotion scheme. It is to be remembered that the promotion scheme, identified with the matrix Q, may in fact be a retraining scheme or some other scheme involving the second attribute used in the rotation planning of the institution. It is assumed that the measure of effectiveness selected to differentiate among the possible alternatives is the total transfer cost within the planning period. With these thoughts, we proceed into the evaluation of the alternatives presented.

H. A METHOD OF COMPARING TWO OR MORE PROPOSED ALTERNATIVES

In this section we develop a method which enables us to compare alternative schemes that satisfy the basic balance equations (2.4) and (2.5). The method focuses on the minimization of the transfer costs between locations incurred during the period for which plans are being made, i.e. the average cost per period in a steady-state model. We assume that the costs incurred in moving an individual of grade k from location i to location j, denoted $c(i,j;k)$, are known.

Now, let $x(i,j;k)$ be the number of individuals of grade k to be transferred from location i to location j during the

period. Let $x(0,j;1)$ be the number of recruits assigned to location j during the period. Here location 0 is meant to indicate an origination point outside the system, and all recruits are assumed to enter the system as members of grade 1. The latter statement is a result of the assumption of a closed institution. We can then formulate the following problem, denoted P1, as a means of comparing proposed alternatives:

$$\text{minimize} \quad z = \sum_{j=1}^{L} c(0,j;1)x(0,j;1)$$

(P1)

$$+ \sum_{i=1}^{L} \sum_{j=1}^{L} \sum_{k=1}^{G} c(i,j;k)x(i,j;k) \qquad (2.17a)$$

subject to

$$\sum_{j=1}^{L} x(0,j;1) = a(0;1), \qquad (2.17b)$$

$$\sum_{j=1}^{L} x(i,j;k) = a(i;k) \ , \ \begin{matrix} i=1,\ldots,L \\ k=1,\ldots,G \end{matrix} \qquad (2.17c)$$

$$\sum_{i=1}^{L} x(i,j;k) = r(j;k) \ , \ \begin{matrix} j=1,\ldots,L \\ k=2,\ldots,G \end{matrix} \qquad (2.17d)$$

and

$$x(0,j;1) + \sum_{i=1}^{L} x(i,j;1) = r(j;1), \ j=1,\ldots,L, \qquad (2.17e)$$

where the $x(i,j;k)$ are restricted to be non-negative. In the above, $a(0;1)$ is seen to be the total number of recruits brought into the system. This is equivalent to the term previously defined as r_o.

Note that the problem P1 is a transportation problem. Further, note that P1 is separable into G transportation

problems, since for any given grade k the constraints may be divided into mutually exclusive subsets of constraints, each containing only variables with third index k. We write out the kth subproblem, denoted P1(k), for k greater than 1:

$$\text{minimize} \quad z(k) = \sum_{i=1}^{L} \sum_{j=1}^{L} c(i,j;k)x(i,j;k) \tag{2.18a}$$

(P1(k))
subject to

$$\sum_{j=1}^{L} x(i,j;k) = a(i;k) \quad i=1,\ldots,L \tag{2.18b}$$

and

$$\sum_{i=1}^{L} x(i,j;k) = r(j;k) \quad j=1,\ldots,L \tag{2.18c}$$

where again the x(i,j;k) are restricted to be non-negative.

For k=1 the above is modified because of the recruits, giving for P1(1):

$$\min \quad z(1) = \sum_{i=0}^{L} \sum_{j=1}^{L} c(i,j;1)x(i,j;1)$$

(P1(1))
subject to

$$\sum_{j=1}^{L} x(i,j;1) = a(i;1) \quad i=0,1,\ldots,L$$

and

$$\sum_{i=0}^{L} x(i,j;1) = r(j;1) \quad j=1,\ldots,L.$$

The problems P1(k) have a solution provided the right-hand sides, or constraint vectors, satisfy

$$\sum_{i=1}^{L} a(i;k) = \sum_{j=1}^{L} r(j;k), \quad k=2,\ldots,G \tag{2.19a}$$

and

$$\sum_{i=0}^{L} a(i;1) = \sum_{j=1}^{L} r(j;1) \qquad (2.19b)$$

Equations (2.19) may be represented in vector notation as

$$\sum_{i=1}^{L} a(i) = \sum_{j=1}^{L} r(j) - (r_o,0,\ldots,0), \qquad (2.19c)$$

where

$$r_o = \sum_{j=1}^{L} x(0,j;1) = a(0;1).$$

Reference to section B quickly confirms that equation (2.19c) is the basic relation which led to equation (2.3) and subsequently to (2.5), one of the basic balance equations.

Assuming now that the institution has the basic structure given in section B, we see that for every combination of B, Q and R satisfying the basic balance equations a feasible solution to P1 exists; consequently, an optimal solution exists, provided only that costs are bounded. This optimal solution will provide a minimum transfer cost for the stated B, Q and R.

The optimal solution to P1 has two elements. One is a cost figure, a scalar representing the summations (2.17a). The other element is the set of numbers of transfers represented by the $x^o(i,j;k)$. Note that the numbers $x^o(i,j;k)$ do not serve to identify the specific individuals to be moved. The solution provides, in gross terms, the numbers of transfers of various types which would take place in a completely controlled environment in order to minimize the costs of the rotation policy under the stated constraints.

We note that due to the separability, the vector $x^o(k)$ which is optimal in P1(k), when combined with all such vectors from the G subproblems, forms a vector x^o which is optimal in P1.

One practical result of solving problem P1 can be shown by a determination of the number of basic variables in the solution to the problem. P1 has 2GL+1 constraints in $L+L^2G$ variables. Of the variables, at most 2GL will appear in the optimal solution at positive levels. In fact, due to the separability, at most G(2L-1)+1 basic variables will be in the solution at positive levels. Thus in an institution with 10 locations and 9 grades, the optimal solution will have no more than 172 of the 910 possible variables that need be considered by the personnel managers. Thus a solution to the transportation problem greatly reduces the types of transfers which must be considered by the personnel manager.

Suppose the manpower planner now wishes to investigate the effect of small changes in elements of the billet structure or rotation rate matrix. We note that under the restrictions of section B the problems P1(k) are related in a complex manner through the dependence of the elements a(i;k) on the requirements at location i for personnel of grades different from k and the matrix Q. That is, for any specific availabilities a and requirements r that satisfy the basic balance equations, P1 will have a solution, and further P1 is completely separable into the subproblems P1(k). However, for a change in either availabilities or requirements of a particular grade, other grades will be affected.

The dependence among elements of the right-hand-sides of problems P1(k) leads to interesting problems. It also points out the desirability of many of the simplifying assumptions made up to this point. Most assumptions made were necessary in order to obtain unique characterizations of the system. Without the assumptions, we are thus unable to obtain uniqueness. Without uniqueness, we are unable to specify the effect of changes in requirements of a particular grade at a particular location on the other elements of the constraint vector to P1.

Given that we are able to uniquely specify the effects of such a change on the system, we wish to determine how these changes affect the optimal cost of P1. With the large number of parameters involved, we suggest some analytical effort be expended prior to effecting the changes. The development of methods to effect this analytical effort is the subject of the next section.

I. SENSITIVITY ANALYSIS OF THE TRANSPORTATION PROBLEM

1. General Discussion

In this section we develop methods which are useful in the detection of "critical" elements among the right-hand-side of constraints (2.17b) through (2.17e). We begin by considering the dual problem to problem P1. We then develop second-order expressions which yield the change in the optimal cost of the transportation problem with changes in the controlled variable when feasibility conditions are taken

into account. Specific expressions are developed for the special cases cited in sections D.1 and D.2 above.

Consider first the dual problem to problem P1, equations (2.17), denoted D1. It is well known from duality theory that the value of the dual variables in the optimal solution to D1 provide a measure of the marginal change to z*, the optimal value of z, with respect to changes in the right hand sides of the constraints in the primal. Thus if u(i;k) is the dual variable associated with the constraint element a(i;k) in the primal problem, then the optimal value of u(i;k), denoted $u^o(i;k)$, gives the change in z* with a unit change in a(i;k) provided the primal basis remains unchanged. Letting u(i;k) have this interpretation and letting v(j;k) be the dual variable associated with the constraint element r(j;k), we can write the dual problem D1 as

$$\text{maximize } d = u(0;1)a(0;1) + \sum_{i=1}^{L} \sum_{k=1}^{G} u(i;k)a(i;k)$$

$$+ \sum_{j=1}^{L} \sum_{k=1}^{G} v(j;k)r(j;k) \tag{2.20a}$$

subject to

$$u(i;1) + v(j;1) = c(i,j;1), \quad \begin{array}{l} i=0,\ldots,L \\ j=1,\ldots,L \end{array} \tag{2.20b}$$

$$u(i;k) + v(j;k) = c(i,j;k), \quad \begin{array}{l} i=1,\ldots,L \\ j=1,\ldots,L \\ k=2,\ldots,G. \end{array} \tag{2.20c}$$

In this problem, the dual variables u and v are unrestricted.

Under the form of Q implied by assumption A2;4 we have

$$a(i;k) = r(i;k)q(k) + r(i;k-1)p(k-1), \quad k=2,\ldots,G$$

and

$$a(i;1) = r(i;1)q(1)$$

for $i=1,\ldots,L$. Substituting these relations into (2.20a), and recalling that assumption A2;3 implies that $r(j;k) = b(j;k)R(j;k)$, we obtain, setting $p(0) = 0$,

$$\begin{aligned} d = u(0;1)a(0;1) &+ \sum_{j=1}^{L}\sum_{k=1}^{G} v(j;k)r(j;k) \\ &+ \sum_{i=1}^{L}\sum_{k=1}^{G} u(i;k)\{r(i;k)q(k) + r(i;k-1)p(k-1)\}. \end{aligned} \tag{2.20a'}$$

The optimal value for d is denoted d* and the values of the dual variables at d* are denoted $u^{o}(i;k)$ and $v^{o}(j;k)$ for all i, j and k.

We note that D1 is separable, just as the primal problem P1 was separable. We write out the first and the kth subproblems

$$\begin{aligned} &\text{maximize } d(1) = u(0;1)a(0;1) + \sum_{j=1}^{L} v(j;1)b(j;1)R(j;1) \\ \text{D1(1)} \quad &\qquad\qquad + \sum_{i=1}^{L} u(i;1)b(i;1)R(i;1)q(1) \\ &\text{subject to } u(i;1) + v(j;1) = c(i,j;1), \quad \begin{array}{l} i=0,\ldots,L \\ j=1,\ldots,L \end{array} \end{aligned} \tag{2.21}$$

and

$$\text{D1}(k) \qquad \begin{aligned} \text{maximize } d(k) = \sum_{i=1}^{L} & u(i;k)\{b(i;k)R(i;k)q(k) \\ & + b(i;k-1)R(i;k-1)p(k-1)\} \\ & + \sum_{j=1}^{L} v(j;k)b(j;k)R(j;k) \end{aligned} \tag{2.22}$$

$$\text{subject to } u(i;k) + v(j;k) = c(i,j;k), \quad \begin{array}{l} i=1,\ldots,L \\ j=1,\ldots,L. \end{array}$$

In the following subsection we use the above sub-problems in determining the effect on the optimal value of the kth subproblem of a change in billets of a selected grade.

2. <u>Changes in a Single Element of B</u>

Suppose the population of personnel of a specific grade, g, at a specific location, ℓ, is increased or decreased. Also suppose that there are no other changes in billets throughout the system. Thus the total population of the institution changes by the same amount. Suppose, as in section D.1 that rotation rates and withdrawal rates are fixed. Thus the billet change is absorbed through a change in the matrix Q.

Recall from Theorem 2.3 that $q(k)$ may be expressed as

$$q(k;\alpha) = \begin{cases} \dfrac{1}{\rho(k)}\{C_1(k) - w(g)R(\ell;g)b(\ell;g)\alpha\}, & k=1,\ldots,g-1 \\[2ex] 1 - w(g) - \dfrac{r_o - \sum_{j=1}^{g} w(j)\rho(j)}{\rho(g) + R(\ell;g)b(\ell;g)\alpha}, & k=g \\[2ex] C_1(k) & k=g+1,\ldots,G. \end{cases}$$

Thus we can observe the effect of a change in $b(\ell;g)$ on $q(k)$ by taking the derivative of the appropriate expressions with respect to α. We thus obtain

$$\frac{\partial q(k;\alpha)}{\partial\alpha} = \begin{cases} -w(g)R(\ell;g)b(\ell;g)\rho(k)^{-1}, & k=1,\ldots,g-1 \\ \dfrac{R(\ell;g)b(\ell;g)\{r_o - \sum_{j=1}^{g} w(j)\rho(j)\}}{\{\rho(g) + R(\ell;g)b(\ell;g)\alpha\}^2}, & k=g \\ 0 & ,k=g+1,\ldots,G. \end{cases} \quad (2.23)$$

With the exception of D1(g), and D1(g+1), the subproblems given by (2.21) and (2.22) can be expressed as functions of α by replacing $q(k)$ with $q(k;\alpha)$ and $p(k)$ with $p(k;\alpha)$. For D1(g) we obtain the optimal value of the objective function, $d^*(g)$, as

$$\begin{aligned} d^*(g) = \sum_{i=1}^{L} & u^o(i;g)\{b(i;g)R(i;g)q(g;\alpha) + \\ & b(i;g-1)R(i;g-1)p(g-1;\alpha)\} \\ & + \sum_{j=1}^{L} v^o(j;g)b(j;g)R(j;g) \\ & + \alpha b(\ell;g)R(\ell;g)\{u^o(\ell;g)q(g;\alpha) + v^o(\ell;g)\}, \end{aligned}$$

and for D(g+1), we obtain the optimal value of the objective function, $d^*(g+1)$, as

$$\begin{aligned} d^*(g+1) = \sum_{i=1}^{L} & u^o(i;g+1)\{b(i;g+1)R(i;g+1)q(g+1;\alpha) + \\ & b(i;g)R(i;g)p(g;\alpha)\} \\ & + \sum_{j=1}^{L} v^o(j;g+1)b(j;g+1)R(j;g+1) \\ & + \alpha b(\ell;g)R(\ell;g)u^o(\ell;g+1)p(g;\alpha). \end{aligned}$$

We note immediately that for k less than g, d*(k) is linear in $q(k;\alpha)$ and $p(k-1;\alpha)$. These are in turn linear in α. Hence for all k less than g, d*(k) is linear in α. We obtain

$$\frac{\partial d^*(k)}{\partial \alpha} = w(g)b(\ell;g)R(\ell;g)\left(\frac{A_k}{\rho(k-1)} - \frac{B_k}{\rho(k)}\right) \quad ,$$

where

$$A_k = \sum_{i=1}^{L} u^o(i;k)b(i;k-1)R(i;k-1)$$

and

$$B_k = \sum_{i=1}^{L} u^o(i;k)b(i;k)R(i;k).$$

For the gth subproblem, we obtain .

$$\frac{\partial d^*(g)}{\partial \alpha} = \{B_g + \alpha b(\ell;g)R(\ell;g)u^o(\ell;g)\}\frac{\partial q(k;\alpha)}{\partial \alpha}$$

$$+ b(\ell;g)R(\ell;g)\{u^o(\ell;g)q(g;\alpha) + v^o(\ell;g)$$

$$+ \frac{A_g w(g)}{\rho(g-1)}\}.$$

Finally, for the g+1st subproblem, we obtain

$$\frac{\partial d^*(g+1)}{\partial \alpha} = -\{A_{g+1} + \alpha b(\ell;g)R(\ell;g)u^o(\ell;g+1)\}\frac{\partial q(g;\alpha)}{\partial \alpha}$$

$$+ b(\ell;g)R(\ell;g)u^o(\ell;g+1)p(g;\alpha).$$

For all subproblems indexed by k greater than g+1, there is no change in the objective function with a change in $b(\ell;g)$. Hence the above partial derivatives express the changes which occur with a change in $b(\ell;g)$.

If the above calculations are carried out for all elements of the billet structure matrix B, the result will

be a display of the possible changes in transportation costs for the various locations and grades. Abnormally high changes for some particular grade, location or grade/location combinations would indicate that the particular community deserves close scrutiny.

J. RELAXATION OF SOME ASSUMPTIONS

1. Relaxation of Assumption A2;3 (Stationarity)

Suppose that we no longer require the requirements $r(i;k;n)$ to be independent of n. Recall that in section B, assumption A2;3 was used to allow the basic balance equations to be expressed in the form of equations (2.4) and (2.5). We now develop a procedure for computationally allowing for the weak form of stationarity where the assumption is that $r(i;k;n) = r(i;k;n-T(i;k))$.

Recall that the above expression was required to obtain relation (2.3),

$$\sum_{i=1}^{L} r(i;n) = r_o(n)N_1, \tag{2.3}$$

and, under this assumption, (2.2) becomes

$$r_o(n) = \sum_{i=1}^{L} \sum_{k=1}^{G} r(i;k;n)w(k). \tag{2.2'}$$

Thus, for the system with "weak" stationarity but not stationarity, (2.2') and (2.3) become the system balance equations.

To proceed, we define $Q(n)$ to be the "promotion" matrix during period n, from which $N_1(n)$ becomes the first row of the fundamental matrix during period n. Thus, under

assumption A2;4 on the structure of Q, a given set of requirements $r(i;k;n)$ and withdrawal rates w uniquely determines $Q(n)$. For those factors which satisfy (2.2') and (2.3), the transportation problem of section H may be defined and the transfer cost determined.

The short-coming of the above is that the relationship between requirements $r(i;k;n)$ and billets $b(i;k)$ is obscured. This is best shown by an example.

Example 2.9. Suppose we have an institution which has specified the tour lengths $T(i;k)$ by

$$T = \begin{pmatrix} 3 & 3 & 3 & 2 & 2 \\ 3 & 3 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 2 & 2 & 1 \end{pmatrix},$$

and has identified the requirements for periods 1 through 6 as

$$r(;;1) = \begin{pmatrix} 100 & 80 & 60 & 40 & 20 \\ 200 & 160 & 120 & 80 & 40 \\ 300 & 240 & 180 & 120 & 60 \\ 400 & 320 & 240 & 160 & 80 \end{pmatrix},$$

$$r(;;2) = \begin{pmatrix} 90 & 70 & 50 & 30 & 15 \\ 210 & 150 & 110 & 70 & 35 \\ 300 & 240 & 180 & 120 & 60 \\ 360 & 280 & 200 & 120 & 80 \end{pmatrix},$$

$$r(;;3) = \begin{pmatrix} 110 & 90 & 70 & 40 & 20 \\ 190 & 145 & 120 & 80 & 40 \\ 300 & 240 & 180 & 120 & 60 \\ 380 & 320 & 240 & 160 & 80 \end{pmatrix},$$

$$r(;;4) = \begin{pmatrix} 100 & 80 & 60 & 30 & 15 \\ 200 & 160 & 110 & 70 & 35 \\ 300 & 240 & 180 & 120 & 60 \\ 400 & 280 & 200 & 120 & 80 \end{pmatrix},$$

$$r(;;5) = \begin{pmatrix} 90 & 70 & 50 & 40 & 20 \\ 210 & 150 & 120 & 80 & 40 \\ 300 & 240 & 180 & 120 & 60 \\ 360 & 320 & 240 & 160 & 80 \end{pmatrix}, \text{ and}$$

$$r(;;6) = \begin{pmatrix} 110 & 90 & 70 & 30 & 15 \\ 190 & 145 & 110 & 70 & 35 \\ 300 & 240 & 180 & 120 & 60 \\ 380 & 280 & 200 & 120 & 80 \end{pmatrix}.$$

The reader may wish to verify that $r(i;k;n) = r(i;k;n-T(i;k))$ for all locations i, grades k and periods n. The reader may also wish to verify that $r(;;7) = r(;;1)$, completing the cycle.

Suppose now that the withdrawal rates are given by $w = (.1\ .3\ .2\ .3\ .4)$. Then we can calculate the $Q(n)$ required to satisfy the balance equations (2.2') and (2.3).

The result is given in Table 2.1. We can also calculate the recruits for each period, i.e. $r_0(n)$. We obtain the vector (660 604 658.5 616 648 614.5).

However, the institution may wish to stabilize the promotion scheme, following the same promotion plan throughout the cycle. One way of determining such a "stabilized" promotion scheme would be to determine the promotion matrix Q based on the billets within the institution and the given rotation scheme. This will be demonstrated through the use of another example.

Example 2.10. For an institution with the structure used in example 2.9, the relation $\sum_{n=m}^{T(i;k)+m-1} r(i;k) = b(i;k)$ gives the billet structure of the institution as

$$B = \begin{pmatrix} 300 & 240 & 180 & 70 & 35 \\ 600 & 455 & 230 & 150 & 75 \\ 300 & 240 & 180 & 120 & 60 \\ 1140 & 600 & 440 & 280 & 80 \end{pmatrix},$$

as may be readily verified. Thus the basic structure of this example is identical with that in example 2.1 and we have a "stabilized" promotion matrix from that example. Using this approach, we expect some disparities between availabilities and requirements in the short run, but would hope that they "average" out over the long run. Table 2.2 gives the excess of availabilities over requirements for the six periods and for all location/pay grade combinations within each period.

$$Q(1) = \begin{pmatrix} .340 & .560 & & & \\ & .300 & .400 & & \\ & & .466 & .333 & \\ & & & .500 & .200 \\ & & & & .600 \end{pmatrix}$$

$$Q(2) = \begin{pmatrix} .371 & .529 & & & \\ & .314 & .386 & & \\ & & .476 & .324 & \\ & & & .476 & .224 \\ & & & & .600 \end{pmatrix}$$

$$Q(3) = \begin{pmatrix} .328 & .572 & & & \\ & .295 & .415 & & \\ & & .472 & .328 & \\ & & & .500 & .200 \\ & & & & .600 \end{pmatrix}$$

$$Q(4) = \begin{pmatrix} .384 & .526 & & & \\ & .321 & .389 & & \\ & & .476 & .324 & \\ & & & .476 & .224 \\ & & & & .600 \end{pmatrix}$$

$$Q(5) = \begin{pmatrix} .325 & .575 & & & \\ & .292 & .408 & & \\ & & .461 & .339 & \\ & & & .500 & .200 \\ & & & & .600 \end{pmatrix}$$

$$Q(6) = \begin{pmatrix} .373 & .527 & & & \\ & .316 & .384 & & \\ & & .482 & .318 & \\ & & & .486 & .224 \\ & & & & .600 \end{pmatrix}$$

Table 2.1. The Promotion Rates for Periods 1 to 6 for Example 2.9.

$$E(;;1) = \begin{pmatrix} -64.6 & -0.9 & -0.2 & -0.7 & 0.4 \\ -129.3 & -1.7 & -0.4 & -1.4 & 0.9 \\ -193.9 & -2.6 & -0.6 & -2.1 & 1.3 \\ -258.6 & -3.5 & -0.8 & -2.8 & 1.7 \end{pmatrix}$$

$$E(;;2) = \begin{pmatrix} -58.2 & 0.6 & 1.1 & 1.1 & 0.3 \\ -135.8 & 10.7 & 0.9 & 0.4 & 0.8 \\ -193.9 & -2.6 & -0.6 & -2.1 & 1.3 \\ -232.7 & 2.4 & 4.6 & 4.4 & -6.7 \end{pmatrix}$$

$$E(;;3) = \begin{pmatrix} -71.1 & -2.3 & -1.6 & 2.6 & 0.4 \\ -122.8 & 3.2 & -6.3 & -1.4 & 0.9 \\ -193.9 & -2.6 & -0.6 & -2.1 & 1.3 \\ -245.6 & -14.4 & -0.8 & -2.8 & 1.7 \end{pmatrix}$$

$$E(;;4) = \begin{pmatrix} -64.6 & -0.9 & -0.2 & 4.4 & 0.3 \\ -129.3 & -1.7 & 4.9 & 0.4 & 0.8 \\ -193.9 & -2.6 & -0.6 & -2.1 & 1.3 \\ -258.6 & -24.3 & 4.6 & 4.4 & -6.7 \end{pmatrix}$$

$$E(;;5) = \begin{pmatrix} -58.2 & 0.6 & 1.1 & -4.0 & 0.4 \\ -135.8 & 10.7 & -4.4 & -1.4 & 0.9 \\ -193.9 & -2.6 & -0.6 & -2.1 & 1.3 \\ -232.7 & -25.4 & -0.8 & -2.8 & 1.7 \end{pmatrix}$$

$$E(;;6) = \begin{pmatrix} -71.1 & -2.3 & -1.6 & 7.7 & 0.3 \\ -122.8 & 3.2 & -1.0 & 0.4 & 0.8 \\ -193.9 & -2.6 & -0.6 & -2.1 & 1.3 \\ -245.6 & 13.3 & 4.6 & 4.4 & -6.7 \end{pmatrix}$$

Table 2.2. The Excesses of Availabilities Over Requirements for Periods 1 through 6 for Example 2.10.

In Table 2.2 we have defined E to be the array of the excesses, where element e(i;k;n) is a(i;k;n) - r(i;k;n). We observe that

$$\sum_{i=1}^{L} \sum_{k=1}^{G} e(i;k;n) = -r_o(n)$$

for all periods n. Referring to section B, we see from the discussion leading up to equation (2.3) that this is to be expected.

We note that the elements of Table 2.2 are relatively small compared with the requirements for each period. To measure the differences between the scheme used in example 2.9, where the promotion matrix was calculated for each period and the scheme used in this example, we let E_9 be the array of excesses in example 2.9 and E_{10} be the array of excesses given in Table 2.2. Then we calculate the percentage differences between the two schemes, denoted P, from

$$p(i;k;n) = \frac{e_{10}(i;k;n) - e_9(i;k;n)}{r(i;k;n)} \times 100.$$

Table 2.3 gives the array P. From P we note that the largest percentage error is approximately 3.8, occurring at location 1 for grade 4 during period 6. Thus at least for this example, the net effect of all the changes in promotion schemes amounts to an increase in ability to meet requirements of only 3.8%.

2. <u>Relaxation of Assumption A2;1 (Single Q)</u>

Suppose now that we assume that there exists a matrix Q_i providing the transformation on past requirements at

$$P(;;1) = \begin{pmatrix} 1.36 & -1.09 & -0.34 & -1.78 & 2.16 \\ 1.36 & -1.09 & -0.34 & -1.78 & 2.16 \\ 1.36 & -1.09 & -0.34 & -1.78 & 2.16 \\ 1.36 & -1.09 & -0.34 & -1.78 & 2.16 \end{pmatrix}$$

$$P(;;2) = \begin{pmatrix} -1.73 & 1.47 & 1.14 & 1.12 & -2.54 \\ -1.73 & 1.67 & 1.11 & 1.13 & -2.54 \\ -1.73 & 1.41 & 1.09 & 1.13 & -2.54 \\ -1.73 & 1.47 & 1.14 & 1.12 & -1.91 \end{pmatrix}$$

$$P(;;3) = \begin{pmatrix} 2.55 & -2.01 & -1.51 & -0.94 & 2.16 \\ 2.55 & -2.23 & -1.42 & -0.96 & 2.16 \\ 2.55 & -2.08 & -1.56 & -0.96 & 2.16 \\ 2.55 & -1.92 & -1.56 & -0.96 & 2.16 \end{pmatrix}$$

$$P(;;4) = \begin{pmatrix} -3.04 & 2.30 & 1.49 & 2.28 & -2.54 \\ -3.04 & 2.30 & 1.68 & 2.07 & -2.54 \\ -3.04 & 2.30 & 1.49 & 2.03 & -2.54 \\ -3.04 & 2.85 & 1.59 & 2.12 & -1.91 \end{pmatrix}$$

$$P(;;5) = \begin{pmatrix} 2.86 & -2.30 & -0.89 & -2.37 & 2.16 \\ 2.86 & -2.63 & -0.69 & -2.62 & 2.16 \\ 2.86 & -2.20 & -0.80 & -2.62 & 2.16 \\ 2.86 & -1.84 & -0.80 & -2.62 & 2.16 \end{pmatrix}$$

$$P(;;6) = \begin{pmatrix} -1.94 & 1.38 & 0.18 & 3.80 & -2.54 \\ -1.94 & 1.56 & 0.21 & 2.98 & -2.54 \\ -1.94 & 1.44 & 0.23 & 2.90 & -2.54 \\ -1.94 & 1.65 & 0.29 & 3.08 & -1.91 \end{pmatrix}$$

Table 2.3. The Percentage Difference Between the Excesses of Availabilities Over Requirements Under the Conditions of Example 2.10 and Example 2.9.

location i giving current availabilities. This would seem to be a "natural" approach to the problem. Differing tour lengths at different locations would seem to result in different promotion opportunities at different locations. We now have

$$a(i) = r(i)Q_i \ , \ i=1,\ldots,L,$$

or for the basic balance equations

$$\sum_{i=1}^{L} a(i) = \sum_{i=1}^{L} r(i)Q_i = \sum_{j=1}^{L} r(j) - (r_o,0,\ldots,0),$$

and

$$\sum_{i=1}^{L} \sum_{k=1}^{G} r(i;k)w(i;k) = r_o,$$

where w(i;k) is a function of location since Q_i is.

Thus the set of promotion matrices Q_i, i=1,...,L, and requirements provides the availabilities and recruits. Unfortunately, much of the analytical work derived in the preceeding is lost to us. For instance, given the requirements r and the withdrawal rates w (now a matrix) we are unable to obtain a unique set of Q_i from the above two relations.

We develop two different approaches to the problem with location-dependent matrices Q_i. First, suppose for each location i we are given a matrix E^i such that $E^i \odot Q = Q_i$, for all i and a common promotion matrix Q. Recall that the symbol $\odot$ implies element by element multiplication. Under assumption A2;4 we have, letting $e^i(j,k)$ be the (j,k)th element of the matrix E^i,

$$r_o = \sum_{i=1}^{L} r(i;k)\{1 - e^i(1,1)q(1,1)\}$$

and

$$0 = \sum_{i=1}^{L} r(i;k)\{1 - e^i(k,k)q(k,k)\}$$

$$- r(i;k-1)e^i(k-1,k)q(k-1,k),$$

the latter statement holding for $k=2,\ldots,G$. These equations can be iteratively solved for the elements $q(m,k)$ of the matrix Q.

The generalization seems to have some merit, but the determination of the matrices E^i may be difficult. Using this in the model for the previous work may lead to results that are heavily dependent on particular E^i used. Since the elements of the E^i do not seem to be directly measureable, we do not pursue this course.

Another approach to the problem with location dependent promotion matrices Q_i is somewhat more general. The straightforward generalization of (2.2) gives

$$r_o(n) = \sum_{i=1}^{L} \sum_{k=1}^{G} r(i;k;n-T(i;k))w(i;k), \quad (2.24)$$

where the modification expresses the withdrawal rate as a function of location as well as grade.

We also have, under the assumption of weak stationarity, that

$$r_o(n) = \sum_{i=1}^{L} \{r(i;1;n) - \sum_{m=1}^{G} r(i;m;n)q_i(m;1)\}, \quad (2.25)$$

and

$$0 = \sum_{i=1}^{L} \{r(i;k;n) - \sum_{m=1}^{G} r(i;m;n)q_i(m;k)\}, \qquad (2.26)$$

for k=2,...,G. Note that in the latter two relations, the elements of the promotion matrix Q_i are indicated by the subscript.

These equations may be manipulated in much the same manner as was used in sections D, E and F above. We provide a sample of the calculations involved. Suppose that the requirements at some location and grade are to be changed. Suppose also that no other requirements are to be affected and that only the promotion matrix for the specified location is to be changed. This would be the case, for instance, where each location in the institution is autonomous insofar as promotions are concerned. Of course, another interpretation would be that if the rotation rate for a particular location/grade were decreased (i.e., the tour length becomes larger), then the promotion rate per tour should increase.

Specifically, let us assume that the requirements at location ℓ for personnel of grade g are to be changed. Let the new requirements be denoted $r'(\ell;g)$ and be given by $r'(\ell;g) = (1+\alpha)r(\ell;g)$. Under the assumption A2;4 of no demotions and at most one promotion per tour, we can calculate the effect of this change on Q_ℓ, since all other Q_i remain fixed.

Let $q'_\ell(k;k) = (1 + \beta(k))q_\ell(k;k)$ for all the diagonal elements of Q_ℓ. We obtain

$$\beta(k) = -\frac{\alpha r(\ell;g)w(\ell;g)}{r(\ell;k)q_\ell(k;k)}, \quad k=1,\ldots,g-1$$

$$\beta(g) = \frac{\alpha q_\ell(g;g+1)}{(1+\alpha)q_\ell(g;g)},$$

and

$$\beta(k) = 0, \qquad k=g+1,\ldots,G.$$

This approach provides an interesting alternative to the approach selected for use in the bulk of the chapter. Computationally, the magnitude of the required storage is seen to be much larger here. Rather than one promotion matrix of size GxG, we now have L matrices of the same size, one for each location. Also, instead of a 1xG vector of withdrawal rates, w, the withdrawal rates are now specified by a LxG matrix.

Preference of one approach over the other seems to be application dependent.

K. EXTENSIONS AND CONCLUSIONS

1. Summary

In the foregoing, we have approached a significant manpower problem - that of the interactive structure between billets, rotation rates and promotions. Under assumptions which were required in order to obtain uniqueness and thus analytical tractability, we have obtained a structure which is versatile and useful in providing insight into the manpower system of a large, multi-location institution.

The versatility of the structure has been demonstrated through several sections which gave some of the different approaches possible to effect a change within the system. A convenient means of evaluating the alternatives presented by the different approaches was given in the form of a transportation problem format. The manner in which various elements affect the cost of the transportation problem was discussed. A method of detecting "critical" elements of the transportation problem was developed.

2. Extensions

There are several possible extensions to the above development. We observe a few of the more significant.

a. Multiple Time Periods

It may well be that current decisions in the rotation scheme will affect future rotations. Each person rotated during this period must rotate at the end of the assigned tour, either out of the system or to a different location. Thus we should be concerned not only with current costs, but should also consider future costs associated with current expenditures. This may be done through a multiple time period transportation problem. It may be desirable to discount the future costs incurred as a result of current decisions.

Such an approach might avert the effects of sub-optimization, i.e. making short-run "optimal" decisions which are costly in the long-run.

b. Restricted Flows

It seems reasonable that many of the possible transfers would not be desirable. For instance, in the U.S. Navy one reason for rotations is the maintenance of equitable periods of sea and shore duty among the personnel within a skill group. In this case a transfer from San Diego-Sea to Norfolk-Sea duty might not be allowed.

In the preceeding discussion this problem was implicitly resolved by assuming that high enough costs $c(i,j;k)$ could be assigned to these transfers to drive them out of the optimal solution. This of course assumes that there is some set of the $x(i,j;k)$ which is feasible and doesn't include any flows restricted by the special considerations. Otherwise, the optimal cost is inflated due to the necessity of remaining feasible and the associated high cost of elements which are restricted.

Hopefully, the use of sensitivity analysis would serve to indicate any violation of such a restricted arc. If in an actual implementation the restrictions on flows do cause problems, it may be worthwhile to consider them explicitly. This may mean that one will have to forego the "nice" structure of the transportation problem.

c. Transient Case

Although much of the above work was involved with the changing of the personnel structure of the institution in various ways, the results were obtained from a steady-state model. Thus the model shows the magnitude of changes

in going from one structure to another. The short-run effect of a change in the system was essentially ignored.

It would be of interest to develop relations, for instance, to show how to "optimally" change the promotion scheme in a sequential nature to get from steady-state under one billet structure to steady-state under another billet structure. Included would be the sequential changes in the billet structure which would be a consequence of the sequentially changing promotion scheme.

Since most large systems probably never acquire steady-state operations, such developments would be helpful in determing how to get from the current personnel structure of the institution to a desired structure.

d. Stochastic Considerations

In the preceeding, it was assumed that the billet structure and rotation rates were given, or alternatively that availabilities and requirements were given. It was also assumed that the system followed the established structure exactly in all cases. In the usual case, only estimates of system parameters are available and the system will fluctuate in a non-deterministic nature about the specified rotation scheme.

For instance, each element a(i;k;n) of the availability vector might be considered to be the result of search of the personnel files (giving the number of personnel of grade k at location i who are to become available during period n) amended by such considerations as the

fraction which will be promoted to k+1 prior to or during period n, the fraction who will leave the system and thus not be available for reassignment and the fraction who will be transferred between the time of the search and the period n. Estimates of these fractions might be obtained through analysis of historical data, examination of legal contracts, promotion policies (current and projected) and/or managerial intutition.

Similarly, requirements are projected based on the results of a search of files, amended by similar considerations. Many factors which are less predictable than those given above influence the actual availabilities and requirements of the system. One of the more significant is in the interrupted tour. A tour is said to be interrupted if the individual concerned rotates in a period other than that period in which the maximum tour length for his present position is completed. There are many reasons for interrupted tours. For instance, an individual may require hospitalization and/or convalescence for a lengthy period of time. This may force the institution to assign the position to another individual. For humanitarian reasons such as death or serious illness in a family, an individual may desire relocation prior to the completion of his present tour. An individual eligible for retirement may decide to retire prior to previously expressed intentions. An employee may be accused of commission of a crime and thus be unavailable while awaiting trial. Someone may die.

The above describes some situations in which tours may be interrupted due to events in the personal lives of employees. Institutional decisions may also cause tours to be interrupted. It may be determined that operations at some location are to be curtailed and personnel reassigned. Some positions may be temporary in nature; with the attainment of the objectives of a particular temporary position, the individual in that position may be reassigned. Similarly, operations may be initiated or expanded. Temporary positions may be created. All of these decisions lead to availabilities and/or requirements which might not be included in the rotation scheme. The early availabilities are directly a cause of interrupted tours. The unforeseen requirements may cause interrupted tours throughout the institution as individuals are selected to fill the requirements.

Actual availabilities and requirements are affected by factors other than interrupted tours. Personnel may transfer between skill groups, possibly maintaining their pay grade. See, for instance, Hayne [21]. These transfers are termed lateral transfers. Also, the institution may use the concept of a "gapped" billet, i.e. a position which exists on paper, but need not be filled at all times. Obviously, gapped billets provide some flexibility in the requirements vector. Also, the transient problem discussed above might leave some dislocations within the system when moving from one steady-state structure to another.

Thus, invalid estimates of parameters, unpredictable fluctuations and changes in institutional policy can lead to dislocations in the rotation scheme. Such dislocations may cause severe variations in the cost of the rotation scheme from that given as a solution to problem P1.

Particular schemes for relating availabilities, requirements and the relations between the two while at the same time accounting for fluctuations might be the subject of significant statistical study. Upon development of the statistical relations, revised estimates (and/or distributional descriptions) could be used in the formulation of a stochastic transportation problem. Since during the period of consideration all availabilities and requirements become known, the structure seems to fit that of a stochastic program with recourse, see for instance Walkup and Wets [36] and Williams [38].

The stochastic nature of the elements of the problem would be applicable if the model were to be used as a forecasting tool. The present use as a planning model can be accomplished quite well without the stochastic considerations.

e. Personnel Planning

The preceeding chapter has been concerned with manpower planning. That is, the idealized system was structured to obtain results in terms of the over-all flows, etc. On the other hand, personnel planning is usually construed to imply concern with the manner in which to manipulate personnel resources in order to best achieve the "goals" of the institution.

By slightly restructuring the above model, it could be useful in personnel planning. For instance, if the availabilities and requirements are taken from a search of the personnel files, then the transportation problem would give the gross numbers of transfers between locations to give the least total transportation costs for the period. Since the optimal solution must be feasible, the solution to the transportation problem also ensures that if personnel are distributed under the scheme in a deterministic system, then all requirements will be met.

Such a structure could also include stochastic elements; thus the comments of the preceeding section apply here also.

3. Conclusions

We see that the structure developed herein has ramifications throughout the manpower and personnel planning of a large multi-location institution. We have been especially concerned with the interaction between billets, rotation rates and promotion schemes as they affect the rotation problem. We have formalized the structure among these elements and provided a means to compare the effects of various changes on the resultant transportation cost of the institution.

Our effort is seen as a beginning, the comments a-e in sub-section 2 above indicating some of the directions that might be undertaken. The pursuit of better and better understanding of the complex interactions between components of the personnel system of large institutions is of increasing

interest. Diseconomies caused by failure to understand these interactions are of concern to both institutional directors and their constituents, both public and private.

III. THE PERSONNEL SELECTION PROCESS WITH TRANSFER COST CONSIDERATIONS

A. INTRODUCTION

In this chapter we develop a model of the personnel selection process within a large institution, where selections are made to determine which individual transfers are to be made. In the previous chapter we determined the gross numbers of transfers of personnel of grade k from location i to location j as the $x(i,j;k)$ in the transportation problem. Here we are concerned with which one of the $a(i;k)$ individuals of grade k who are available to be transferred from location i will be sent to location j to satisfy a requirement among the $r(j;k)$. Hopefully, the selections will be made so that the total number of transfers during the planning period will approximate the $x^{o}(i,j;k)$ from the previous model.

The process to be modelled has as its principal concern the finding of a qualified replacement to fill a single given demand. Here qualified might mean satisfactory performance in prerequisite positions, or it might mean attainment of a specific pay grade, or it might mean just finding a "body" to fill a slot. The degree of qualification will not be specified further.

The general approach taken is to observe the behavior of an idealized system. The idealized system to be studied is restricted in scope and application, but lends itself to

providing insight into the structure of the problem. Later sections generalize the results somewhat.

The first system we study we term the "Ehrenfest Decision Model." Here we draw on the results of the well-known Ehrenfest Urn model of heat flow between isolated bodies (for instance, see Feller [9]). The known results are adapted into a decision-making framework. In the special case of the Ehrenfest Urn problem, one becomes concerned with the choice set over which decisions are to be made. Results are obtained in the form of probability distributions over the choice set. From this, distributional results describing the transfer among locations are obtained.

We develop some approximations which are helpful in that they provide insight into significant parameters of the system.

The Ehrenfest Urn model *per se* is usually considered as consisting of two isolated bodies. The Ehrenfest Decision Model used here is developed for the case of two locations, then generalized to three locations. Further, the choice set is generalized to include cases where sequential decisions are allowed.

A more general type of generalization is a description of the time dependence of the distributions. This leads directly to the use of diffusion approximations.

B. THE BASIC MODEL -- THE TWO LOCATION PROBLEM

1. Ehrenfest Decision Model

Consider a closed firm or institution. The concept of a closed institution and the various reasons for transfers between the locations in a multi-location institution were discussed in section A of chapter II above. Suppose now that the institution has only two locations where operations are conducted.

As in chapter II, we assume that personnel are described by two attributes. Here we use current location and qualification as the two attributes of concern. Thus it may be that we are restricted to personnel of a single skill category/pay grade combination. We may require also that peripheral qualifications be evidenced within the restricted group of personnel to be considered in the selection process.

We assume the existence of a central decision making organization, commonly called an *assignment center* or *personnel distribution center*, whose function is to determine which individual to assign in response to a requirement or demand.

We call the two locations R (for red) and B (for black). We call the personnel distribution center M* (for manager). Demands made to M* for personnel originate at R or B. They also include a specification of the qualification required to perform the duties of the position. In the context of chapter II, demands occur due to forthcoming completions of tours, tours being interrupted, or simply creation of new positions.

Costs are assumed to be incurred whenever a replacement, i.e. a person selected for transfer in response to the demand, is currently in the location other than the one originating the demand. In this sense, a physical separation of the locations is assumed to exist. The institution is assumed to be liable for the expenses incurred in moving the replacement, his family and their possessions from his current location to the site of his next position.

Once a demand is received at the personnel distribution center M*, it may take considerable time and effort to locate a suitable replacement for the individual currently in the position. The assignment center must find an individual who is now, or soon will be, available for reassignment and who meets the specified qualifications.

The assessment of qualification may include an appraisal of each individual's past assignments and performance. The purpose of the rotation plan must be considered. If, for instance, one of the firm's goals for the rotation program is to provide exposure to varying aspects of the firm's operations, the assignment center would not want to assign someone to a position he has previously held. Other factors might be important. The promotional opportunity provided by a particular position may correlate with the promotional potential of individuals assigned that position. That is, a "high exposure" position may only be given to one of a select group within the broad classification being considered.

Some theoretical consideration of the time to find a suitable replacement when searching a list of potentially qualified personnel is given in Chapter IV.

Upon determination of a replacement, the person currently in the position for which the demand was generated is assumed to become available for reassignment. Thus his personnel file is added to the group which is searched by the distribution center upon receipt of a demand. Thus every exit from the group available for reassignment is matched by an entry into the group.

Here we ignore any interactions between groups available for reassignment, i.e. groups of different skill types or pay grades. Thus the size of the population available for reassignment is assumed fixed.

We construct a simple model reflecting the structure developed above. Suppose the records of those personnel available for reassignment are considered to be balls in a "decision" urn M*, where M denotes the fixed total number of balls in the urn. The balls in M* are painted red or black, depending on the current physical location of the individual whose file is represented by the ball.

Let R(n) denote the number of red balls in urn M* following the nth transfer to occur in the system. Similarly B(n) denotes the number of black balls in M* at the same point in time. Clearly, R(n) + B(n) = M for all n.

Demands may be considered to be the outcome of independent Bernoulli trials. This is reasonable in the sense

that if there are N_R positions of a certain type at location R, and N_B positions of the same type at location B, then the fraction of demands originating from R should approximate $N_R/(N_R + N_B)$. We also consider demands to occur independently, and let r be the probability that a random demand originates from location R, where

$$r = \frac{N_R}{N_R + N_B} .$$

We also let b = 1-r, i.e. the probability that the demand originates from location B.

Graphically, the model may be represented as Figure 3.1.

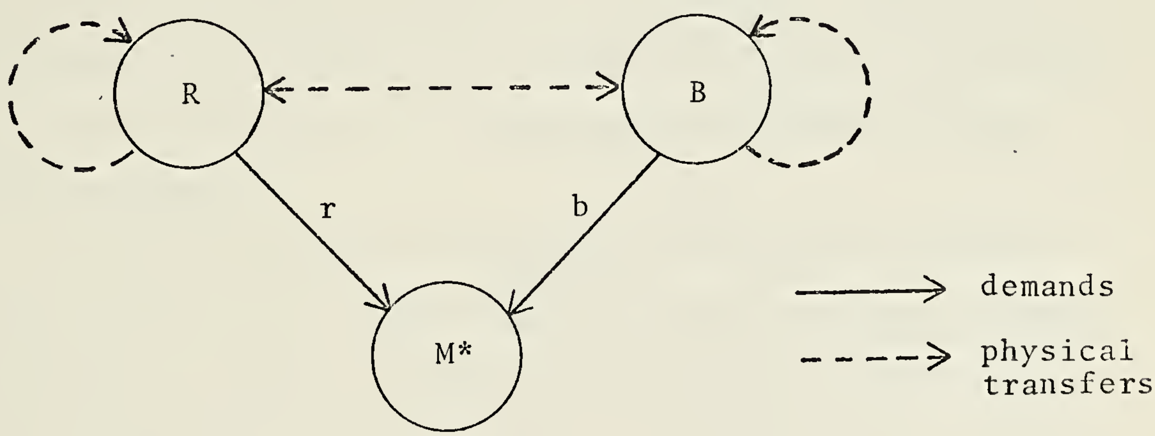


Figure 3.1. The Two Location Ehrenfest Decision Model.

In the next section, we obtain results on the steady-state probability distribution of the composition of urn M*.

2. Distributional Results Under a Simple Decision Rule

Suppose the personnel distribution center M* simply finds the first qualified person and effects a transfer. In the simplest case where all personnel whose files are represented by balls in M* are equally qualified for all demands, this would imply that the policy is to take a random draw from urn M* and effect a transfer.

In more complex situations, there may be a lengthy search to find qualified personnel. In the latter case, the search would terminate upon the location of the first person qualified for the position from which the demand originated.

In this section, the steady-state probability distribution of $R(n)$ is to be determined. The value of $R(n)$ at any time n is called the state of the system. Also to be determined is the steady-state probability that an arbitrary move is a "cost" move, i.e. represents a transfer between the two locations.

Let $R(\infty)$ denote the number of red balls in urn M* after an arbitrarily large number of transfers have occurred. Thus $R(\infty)$ is a random variable describing the number of red balls in urn M* once the system is in steady-state. Thus we wish to find the probability distribution of $R(\infty)$. Note that determination of the probability distribution of $R(\infty)$ also implies the determination of the probability distribution of $B(\infty)$.

Theorem 3.1. Under the above conditions, $R(\infty)$ is distributed Binomially with parameters M and r, i.e. $R(\infty) \sim \text{Bin}(M,r)$

Before proceeding with the proof of theorem 3.1, it is useful to develop some notation. Let $\Pr\{A = a\}$ be the probability that the discrete random variable A assumes the value a. Let $\Pr\{A = a \mid B = b\}$ be the conditional probability that the discrete random variable A assumes the value a given that the discrete random variable B assumes the value b.

Proof: Consider the conditional probability statements:

$$\Pr\{R(n) = j \mid R(n-1)=j-1\} = \frac{M-(j-1)}{M}\, r,$$

$$\Pr\{R(n) = j \mid R(n-1)=j\} = \frac{j}{M}\, r + \frac{M-j}{M}(1-r) \qquad (3.1)$$

and

$$\Pr\{R(n) = j \mid R(n-1)=j+1\} = \frac{j+1}{M}(1-r),$$

where the first equation states that if $R(n-1) = j-1$, $R(n) = j$ if and only if a demand arose at R (an occurrence with probability r) and a black ball was chosen from urn M* (an occurrence with probability $1-(j-1)/M$). Similar interpretations apply to the other two expressions. Note that changes of more than one unit are excluded.

Now let $\theta(z)$ be the probability generating function, i.e.

$$\theta(z;n) = E\{z^{R(n)}\} = \sum_{j=0}^{M} z^j \Pr\{R(n) = j\}, \qquad (3.2)$$

where $E\{A\}$ is the expected value of the random variable A. At steady-state, $\Pr\{R(n)=j\} = \Pr\{R(n-1)=j\}$. Let $p(j)$ denote $\Pr\{R(\infty)=j\}$. Assuming steady-state, we remove the condition on $R(n-1)$ in equations (3.1), obtaining

$$\theta(z;\infty) = \sum_{j=0}^{M} z^j \left\{ \frac{M-(j-1)}{M} rp(j-1) + \frac{j+1}{M}(1-r)p(j+1) \right.$$
$$\left. + \left\{ \frac{jr}{M} + \frac{(M-j)}{M}(1-r) \right\} p(j) \right\},$$

or

$$\frac{d\theta(z;\infty)}{dz} - \frac{Mr}{1-r+rz}\,\theta(z;\infty) = 0.$$

The solution to the above differential equation subject to $\theta(1;\infty)=1$ is

$$\theta(z;\infty) = (1-r+rz)^M.$$

Hence $R(\infty)$ is distributed binomially with parameters M and r.

The next question of concern is that of the probability of an arbitrary move being one between locations. Let $c(n|j)$ be the probability that an inter-location transfer occurs on the nth transfer given that $R(n-1)=j$. We obtain

$$c(n|j) = \frac{j}{M}(1-r) + \frac{M-j}{M}r$$
$$= r + \frac{j}{M}(1-2r).$$

At steady-state the unconditioning in straightforward and we obtain, letting $c(\infty)$ be the probability that an arbitrary move is an inter-location move when the system is in steady state,

$$c(\infty) = 2r(1-r).$$

Thus if C is the cost of transfer between location R and B, the expected cost of satisfying an arbitrary demand when the system is in steady-state is $2r(1-r)C$.

In the next section, we model a more complex system, where two or more qualified personnel are selected prior to a decision being made as to which one to transfer. Distributional results are obtained for the steady-state case.

3. Variable Length Selection List (L)

Suppose now that the personnel distribution center is required to find a specified number of qualified personnel prior to effecting a transfer. Denote by L the number specified. Ignoring momentarily the problems associated with the search time to find the L specified people, we analyze the Ehrenfest Decision Model in a manner similar to the above. In this section, however, the state-dependent Markovian properties of the system are useful in the analysis.

The transfer policy specified for this section is that if any of the L selected personnel are currently located at the location from which the demand originated, then one of those is transferred to the new position and the remaining L-1 personnel files are returned to M*. Hence an inter-location transfer occurs only when all L selectees are currently in the location other than the one from which the demand originated.

It is obvious that some restrictions exist on L. For instance, if L were larger than M, the size of the

population in M*, then the list would never be completed. The relationship between L and M is now investigated.

Theorem 3.2. (a) If the selection list has length less than or equal to $1+\frac{M}{2}$, then the system in steady-state is absorbed in an absorbing class with M-2L+3 states.

(b) If the selection list has length greater than or equal to $1+\frac{M}{2}$, then the system steady-state is absorbed in one of 2L-M-1 absorbing states.

Proof. (a) $L \leq 1+\frac{M}{2}$ implies that $L-1 \leq M-L+1$. If $R(n) \leq L-1$ then $R(n+1) \geq R(n)$. That is, each selection of L balls when $R(n) \leq L-1$ must have at least one black ball. Hence under the transfer policy stated above, for R(n) in the specified range no red ball will ever be transferred to B. Hence R(n+1) can never be less than R(n).

If $R(n) \geq M-L+1$, then $B(n) \leq L-1$. By argument similar to the above, $B(n+1) \geq B(n)$. Hence $R(n+1) \leq R(n)$.

Furthermore, in each case there is some positive probability that the inequality will be strict. For $R(n) \leq L-1$,

$$\Pr\{R(n+1) > R(n) \mid R(n)=j, j \leq L-1\} = \frac{\binom{M-j}{L}}{\binom{M}{L}} r,$$

where $\binom{A}{B}$ denotes the combinatorial, i.e. $\binom{A}{B} = \frac{A!}{B!(A-B)!}$ for A greater than or equal B. For $R(n) \geq M-L+1$, we obtain

$$\Pr\{R(n+1) < R(n) \mid R(n)=j, j \geq M-L+1\} = \frac{\binom{j}{L}}{\binom{M}{L}} (1-r).$$

Hence R(∞) may take on values only between L-1 and M-L+1. Furthermore, it is readily seen that for $L \leq R(n) \leq M-L$, there is positive probability that $R(n+1) = R(n)$, that $R(n+1) = R(n)+1$ and that $R(n+1) = R(n)-1$. Hence all values from L-1 to M-L+1 are in the range of R(∞). Hence the system is absorbed in an absorbing class with M-L+1-(L-2) or M-2L+3 states.

(b) $L \geq 1+\frac{M}{2}$ implies $L-1 \geq M-L+1$. If $R(n) \geq M-L+1$ for any n, then the argument above gives us that $R(n+1) \leq R(n)$. Also, if $R(n) \leq L-1$ we have that $R(n+1) \geq R(n)$. Hence for $M-L+1 \leq R(n) \leq L-1$ we have that $R(n+1) = R(n)$. Hence each state from M-L+1 to L-1 is an absorbing state and we have L-1-(M-L) or 2L-M-1 absorbing states.

Example 3.1. Let M=11, L=5. The one step transition matrix is represented by

$$
\begin{array}{c|cccccccccccc}
R(n) \backslash R(n+1) & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\hline
0 & x & x & & & & & & & & & & \\
1 & & x & x & & & & & & & & & \\
2 & & & x & x & & & & & & & & \\
3 & & & & x & x & & & & & & & \\
4 & & & & & x & x & & & & & & \\
5 & & & & & x & x & x & & & & & \\
6 & & & & & & x & x & x & & & & \\
7 & & & & & & & x & x & & & & \\
8 & & & & & & & & x & x & & & \\
9 & & & & & & & & & x & x & & \\
10 & & & & & & & & & & x & x & \\
11 & & & & & & & & & & & x & x
\end{array}
$$

where the x's represent positive probabilities. To trace an evolution towards steady-state, suppose R(n) = 2. Then R(n+1) can take on values 2 or 3. If it assumes the value 2, the next transfer is under the same probabilistic structure. If

$R(n+1) = 3$, then $R(n) > 2$ for all $m \geq n+1$, by the above theorem. The system continues operation until $4 \leq R(n) \leq 7$ for some m. (Obviously in this case, the absorbing class is entered whenever $R(n) = 4$ for the first time.) Once in the absorbing class, a probability distribution can be determined which will describe the state of the system at an arbitrary time in the future (i.e. after many transitions have occurred). We do this.

For the rest of this chapter, L is considered to be less than $1 + \frac{M}{2}$. This is the case of interest in that it lends itself to the determination of a steady-state probability distribution. The other case discussed above is obviously dependent on initial conditions and hence is of limited interest.

We write out the conditional probability statements comparable to equations (3.1) for the case of variable length selection lists:

$$\Pr \{ R(n)=j \mid R(n-1)=j-1 \} = \frac{\binom{M-(j-1)}{L}}{\binom{M}{L}} r,$$

$$\Pr \{ R(n)=j \mid R(n-1)=j \} = 1 - \frac{\binom{j}{L}}{\binom{M}{L}} (1-r) - \frac{\binom{M-j}{L}}{\binom{M}{L}} r, \quad (3.3)$$

and

$$\Pr \{ R(n)=j \mid R(n-1)=j+1 \} = \frac{\binom{j+1}{L}}{\binom{M}{L}} (1-r).$$

Again, the possibility of other transitions is excluded.

From the proof of theorem 3.2, we know that states $0,1,\ldots,L-2$ and states $M-L+2,\ldots,M$ are transient. To determine the steady-state distribution of M^*, we solve

$$\pi = \pi P,$$

where π is a $1\times(M+1)$ vector with elements $\pi(j) = \Pr\{R(\infty)=j\}$ for $j=0,1,\ldots,M$, and P is the one step transition matrix with elements $p_{ij} = \Pr\{R(n)=j \mid R(n-1)=i\}$. We have that

$$\pi(0) = \cdots = \pi(L-2) = \pi(M-L+2) = \cdots = \pi(M) = 0, \text{ and}$$

$$\sum_{j=L-1}^{M-L+1} \pi(j) = 1.$$

We solve for the $\pi(j)$'s recursively, first obtaining an expression for each $\pi(j)$ in terms of $\pi(L-1)$ as

$$\pi(L+k) = \left(\frac{r}{1-r}\right)^{k+1} \frac{\prod_{j=0}^{k} \binom{M-L+1-j}{L}}{\prod_{j=0}^{k} \binom{L+j}{L}} \pi(L-1), \quad k=0,\ldots,M-2L+1.$$

Letting $A(k) = \prod_{j=0}^{k} \binom{M-L+1-j}{L}$ and $C(k) = \prod_{j=0}^{k} \binom{L+j}{L}$, the above relation is written as

$$\pi(L+k) = \left(\frac{r}{1-r}\right)^{k+1} \frac{A(k)}{C(k)} \pi(L-1). \tag{3.4}$$

But we also have that $\sum_{j=L-1}^{M-L+1} \pi(j) = \pi(L-1) + \sum_{j=0}^{M-2L+1} \pi(L+j) = 1.$

Hence, substituting (3.4) into the above,

$$\pi(L-1) = \left(1 + \sum_{k=0}^{M-2L+1} \left(\frac{r}{1-r}\right)^{k+1} \frac{A(k)}{C(k)}\right)^{-1}.$$

Hence,

$$\pi(L+j) = \frac{\left(\frac{r}{1-r}\right)^{j+1} \frac{A(j)}{C(j)}}{1 + \sum_{k=0}^{M-2L+1} \left(\frac{r}{1-r}\right)^{k+1} \frac{A(k)}{C(k)}}, \quad j=0,\ldots,M-2L+1. \tag{3.5}$$

The preceeding relation is not in a computationally efficient form. Letting $D(j) = A(j)/C(j)$ we see

$$D(0) = \frac{\binom{M-L+1}{L}}{\binom{L}{L}}, \tag{3.6a}$$

and

$$D(1) = \frac{\binom{M-L}{L}}{\binom{L+1}{L}} D(0).$$

In general we can write

$$D(j) = \frac{\binom{M-L+1-j}{L}}{\binom{L+j}{L}} D(j-1), \quad j=0,\ldots,M-2L+1,$$

or

$$D(j) = \frac{(M-L+1-j)!\ j!}{(M-2L+1-j)!\,(L+j)!} D(j-1). \tag{3.6b}$$

<u>Lemma 3.3.</u> $D(M-2L-j) = D(j), \quad j=0,1,\ldots, \left[\frac{M+1}{2}\right] - L,$ where [ ] indicates "greatest integer less than or equal" the value indicated.

Proof. By straightforward substitution into (3.6), we have

$$D(0) = \frac{(M-L+1)!\ 0!}{(M-2L+1)!\ L!},$$

and

$$D(M-2L+1) = \frac{L!\,(M-2L+1)!}{0!\;(M-L+1)!}\,D(M-2L).$$

By evaluating A(M-2L+1) and C(M-2L+1), we see that D(M-2L+1) = 1. Hence

$$D(M-2L) = \frac{0!\,(M-L+1)!}{(M-2L+1)!\;L!} = D(0).$$

Hence the lemma is true for j=0. Assume the lemma is true for some j within the allowable range, i.e. $j=0,\ldots,\frac{M-1}{2}+1$. We show that the lemma holds for j+1.

$$D(j+1) = \frac{(M-L-j)!\,(j+1)!}{(M-2L-j)!\;(L+j+1)!}\,D(j)$$

$$= \frac{(M-L-j)!\,(j+1)!}{(M-2L-j)!\;(L+j+1)!}\,D(M-2L-j).$$

But from (3.6b)

$$D(M-2L-j) = \frac{(L+j+1)!\,(M-2L-j)!}{(j+1)!\,(M-L-j)!}\,D(M-2L-j-1).$$

By transposing, we see that

$$D(M-2L-(j+1)) = D(j+1).$$

Hence the lemma is shown by induction.

The significance of the lemma is in the reduction of the number of coefficients D(j) which must be calculated in order to determine the steady state probabilities from equation (3.5).

Calculations to determine the $\pi(L+j)$'s are straightforward once the D(j)'s are computed. A FORTRAN program was

to do the computations. A sample of the results for M=20, for L=1,..., $\frac{M}{2}+1$, and for r=.1,.2,...,.5 are given in Tables 3.1 and 3.2.

We note that for L=1 the values in Tables 3.1 and 3.2 are binomial which agrees with the theoretical results of section B.2 above. In Table 3.1, the effect on the steady-state distribution of urn M* resulting from an increase in L is shown. Such a table would be of interest for a stable system with a known r. If the system under study were to contemplate changes in the personnel structure, then Table 3.2 would show for a given L the effect of a change in r on the steady-state distribution of M*.

The next section develops some approximations to the calculations of this section. These approximations are used to determine the effect of changing L on the steady-state probability of incurring a cost move.

4. Approximations to the Steady-State Probabilities of the Variable List Length Process

In this section we develop some approximations to the steady-state probabilities developed in the last section. Comparison values are computed for the same conditions under the procedure of the last section and under the approximation.

If L is much less than M, as seems reasonable in many cases, then one might treat the model as selection with replacement. Under selection with replacement, the one step transition probabilities are based on powers of the fractional

M = 20   r = .100

LIST LENGTH =		1	2	3	4	5	6	7	8	9	10	11
R(∞) =	0	.122										
	1	.270	.001									
	2	.285	.015									
	3	.190	.088	.001								
	4	.090	.221	.018								
	5	.032	.294	.113	.010							
	6	.009	.229	.285	.103	.016	.001					
	7	.002	.110	.331	.326	.172	.055	.010	.001			
	8		.034	.188	.370	.440	.372	.235	.103	.025		
	9		.007	.055	.161	.306	.455	.573	.631	.601	.448	
	10		.001	.008	.028	.062	.111	.175	.257	.367	.546	1.00
	11			.001	.002	.004	.006	.007	.008	.007	.006	

M = 20   r = .200

		1	2	3	4	5	6	7	8	9	10	11
R(∞) =	0	.012										
	1	.058										
	2	.137	.001									
	3	.205	.008									
	4	.218	.046	.001								
	5	.175	.139	.018	.001							
	6	.109	.243	.098	.019	.002						
	7	.055	.263	.255	.136	.049	.012	.002				
	8	.022	.183	.326	.347	.280	.181	.094	.035	.007		
	9	.007	.084	.214	.341	.440	.500	.517	.487	.408	.262	
	10	.002	.026	.073	.134	.201	.275	.355	.448	.560	.722	1.00
	11		.005	.013	.021	.027	.031	.032	.030	.025	.016	
	12		.001	.001	.001	.001	.001					

Table 3.1.

M = 20 r = .300

LIST LENGTH		1	2	3	4	5	6	7	8	9	10	11
R(∞)	= 0	.001										
	1	.007										
	2	.028										
	3	.072	.001									
	4	.130	.008									
	5	.179	.039	.003								
	6	.192	.117	.028	.004							
	7	.164	.217	.127	.052	.016	.003					
	8	.114	.260	.278	.227	.155	.090	.043	.015	.002		
	9	.065	.204	.313	.382	.419	.425	.404	.357	.282	.170	
	10	.031	.107	.184	.257	.328	.401	.478	.561	.664	.799	1.00
	11	.012	.037	.057	.070	.077	.078	.074	.066	.052	.031	
	12	.004	.009	.009	.008	.005	.003	.001	.001			
	13	.001	.001	.001								

M = 20 r = .400

		1	2	3	4	5	6	7	8	9	10	11
R(∞)	0											
	1											
	2	.003										
	3	.012										
	4	.035	.001									
	5	.075	.008									
	6	.124	.039	.007	.001							
	7	.166	.113	.052	.019	.005	.001					
	8	.180	.209	.176	.127	.080	.044	.020	.007	.001		
	9	.161	.255	.308	.332	.337	.325	.298	.255	.195	.114	
	10	.117	.208	.281	.347	.412	.477	.546	.624	.717	.835	1.00
	11	.071	.114	.136	.147	.150	.144	.132	.113	.087	.051	
	12	.035	.041	.035	.025	.016	.009	.004	.001			
	13	.015	.010	.005	.002							
	14	.005	.002									
	15	.001										

Table 3.1.

M = 20 r = .500

LIST LENGTH=	1	2	3	4	5	6	7	8	9	10	11
R(∞) = 0											
1											
2											
3	.001										
4	.005										
5	.015	.001									
6	.037	.009	.002								
7	.074	.041	.017	.006	.002						
8	.120	.113	.089	.062	.038	.021	.009	.003	.001		
9	.160	.209	.232	.242	.240	.228	.207	.175	.133	.077	
10	.176	.254	.320	.380	.440	.502	.568	.644	.732	.846	1.00
11	.160	.209	.232	.242	.240	.228	.207	.175	.133	.077	
12	.120	.113	.089	.062	.038	.021	.009	.003	.001		
13	.074	.041	.017	.006	.002						
14	.037	.009	.002								
15	.015	.001									
16	.005										
17	.001										

Table 3.1. The steady state probabilities for the two location variable length selection list problem, M=20. For the specified probability that an arbitrary demand arises at location R, the probabilities are given as a function of the length of the selection list, L.

M = 20 L = 1

	r =	.1	.2	.3	.4	.5	
R(∞) =	0	.122	.012	.001			20=B(∞)
	1	.270	.058	.007			19
	2	.285	.137	.028	.003		18
	3	.190	.205	.072	.012	.001	17
	4	.090	.218	.130	.035	.005	16
	5	.032	.115	.179	.075	.015	15
	6	.009	.109	.192	.124	.037	14
	7	.002	.055	.164	.166	.074	13
	8		.022	.114	.180	.120	12
	9		.007	.065	.161	.160	11
	10		.002	.031	.117	.176	10
	11			.012	.071	.160	9
	12			.004	.035	.120	8
	13			.001	.015	.074	7
	14				.005	.037	6
	15				.001	.015	5
	16					.005	4
	17					.001	3

M = 20 L = 2

R(∞) =	1	.001					19=B(∞)
	2	.015	.001				18
	3	.088	.008	.001			17
	4	.221	.046	.008	.001		16
	5	.294	.139	.039	.008	.001	15
	6	.229	.243	.117	.039	.009	14
	7	.110	.263	.217	.113	.041	13
	8	.034	.183	.260	.209	.113	12
	9	.007	.084	.204	.255	.209	11
	10	.001	.026	.107	.208	.254	10
	11		.005	.037	.114	.209	9
	12		.001	.009	.041	.113	8
	13			.001	.010	.041	7
	14				.002	.009	6
	15					.001	5

M = 20 L = 3

R(∞)	2						18=B(∞)
	3	.001					17
	4	.018	.001				16
	5	.113	.018	.003			15
	6	.285	.098	.028	.007	.002	14
	7	.331	.255	.127	.052	.017	13
	8	.188	.326	.278	.176	.089	12
	9	.055	.214	.313	.308	.232	11
	10	.008	.073	.184	.281	.320	10
	11	.001	.013	.057	.136	.232	9
	12		.001	.009	.035	.089	8
	13			.001	.005	.017	7
	14					.002	6

Table 3.2.

M = 20   L = 4

		r = .1	.2	.3	.4	.5	
R(∞) =	3						17=B(∞)
	4						16
	5	.010	.001				15
	6	.103	.019	.004	.001		14
	7	.326	.136	.052	.019	.006	13
	8	.370	.347	.227	.127	.062	12
	9	.161	.341	.332	.332	.242	11
	10	.028	.134	.257	.347	.380	10
	11	.002	.021	.070	.147	.242	9
	12		.001	.008	.025	.062	8
	13				.002	.006	7

M = 20   L = 5

		.1	.2	.3	.4	.5	
R(∞) =	4						16=B(∞)
	5						15
	6	.016	.002				14
	7	.172	.049	.016	.005	.002	13
	8	.440	.280	.155	.080	.038	12
	9	.306	.440	.419	.337	.240	11
	10	.062	.201	.328	.412	.440	10
	11	.004	.027	.077	.150	.240	9
	12		.001	.005	.016	.038	8
	13					.002	7

M = 20   L = 6

		.1	.2	.3	.4	.5	
R(∞) =	5						15=B(∞)
	6	.001					14
	7	.055	.012	.003	.001		13
	8	.372	.181	.090	.044	.021	12
	9	.455	.500	.425	.325	.228	11
	10	.111	.275	.401	.477	.502	10
	11	.006	.031	.078	.144	.228	9
	12		.001	.003	.009	.021	

M = 20   L = 7

		.1	.2	.3	.4	.5	
R(∞) =	6						14=B(∞)
	7	.010	.002				13
	8	.235	.094	.043	.020	.009	12
	9	.573	.517	.404	.298	.207	11
	10	.175	.355	.478	.546	.568	10
	11	.007	.032	.074	.132	.207	9
	12			.001	.004	.009	8

M = 20   L = 8

		.1	.2	.3	.4	.5	
R(∞) =	7	.001					13=B(∞)
	8	.103	.035	.015	.007	.003	12
	9	.631	.487	.357	.255	.175	11

Table 3.2.

M = 20 L = 8 continued

r =		.1	.2	.3	.4	.5	
R(∞)	= 10	.257	.448	.561	.624	.644	10=B(∞)
	11	.008	.030	.066	.113	.175	9
	12			.001	.001	.003	8
	13						

M = 20 L = 9

R(∞)	8	.025	.007	.002	.001	.001	12=B(∞)
	9	.601	.408	.282	.195	.133	11
	10	.367	.560	.664	.717	.732	10
	11	.007	.025	.052	.087	.133	9
	12					.001	8

M = 20 L = 10

R(∞)	9	.448	.262	.170	.114	.077	11=B(∞)
	10	.546	.722	.799	.835	.846	10
	11	.006	.016	.031	.051	.077	9

M = 20 L = 11

R(∞)	10	1.00	1.00	1.00	1.00	1.00	10=B(∞)

Table 3.2. The steady-state probabilities for the two location variable length selection list problem, M = 20. For the specified length of selection list, L, the probabilities are given as functions of r, the probability that an arbitrary demand arises at location R.

composition of urn M* rather than on the combinatorials encountered in the last section. Equations (3.3) are approximated by

$$\Pr\{R(n)=j\,|\,R(n-1)=j-1\} = \left(\frac{M-(j-1)}{M}\right)^L r, \qquad (3.3a)$$

$$\Pr\{R(n)=j\,|\,R(n-1)=j\} = 1 - \left(\frac{j}{M}\right)^L (1-r) - \left(\frac{M-j}{M}\right)^L r,$$

and $$\Pr\{R(n)=j\,|\,R(n-1)=j+1\} = \left(\frac{j+1}{M}\right)^L (1-r).$$

Letting $p(j;n) = \Pr\{R(n)=j\}$, we obtain

$$p(j;n) = \left(\frac{M-j+1}{M}\right)^L rp(j-1;n-1) + \left(\frac{j+1}{M}\right)^L (1-r)p(j+1;n-1)$$

$$+ \left(1 - \left(\frac{j}{M}\right)^L (1-r) - \left(\frac{M-j}{M}\right)^L r\right)p(j;n-1).$$

The above equation is representative of a system of equations which are indexed on the states j and time periods n. These equations are quite messy. A simple deterministic version, suggested by Gaver, may be obtained by letting $\rho(t)$ denote the number of red balls in M* at time t. Time is considered to be measured in units of transitions. Then we can express the time rate of change of $\rho(t)$ in terms of $\rho(t)$. The fraction of red balls in urn M* at time t will increase if L black balls are chosen in response to a demand from R, and will decrease if L red balls are chosen in response to a demand from B. Otherwise, the fraction of red balls, and thus the fraction of black balls, will remain the same. Thus the time rate of change of $\rho(t)$ may be expressed by

$$\frac{d\rho(t)}{dt} = \left(\frac{M-\rho(t)}{M}\right)^L r - \left(\frac{\rho(t)}{M}\right)^L (1-r). \tag{3.7}$$

At steady-state, denoted $\rho = \rho(\infty)$, $\frac{d\rho}{dt} = 0$, which implies, letting $f = \rho/M$ denote the expected value of the fraction of red balls in urn M* at steady-state,

$$(1-f)^L r = f^L(1-r).$$

Thus the expected value of the fraction of red balls in urn M* is

$$f = \frac{r^{1/L}}{r^{1/L} + (1-r)^{1/L}}. \tag{3.8}$$

Thus the quantity Mf should approximate the expected number of red balls in M* at steady-state. We note that for L=1, the approximation is exact.

Table 3.3 gives values for the expected number of red balls in M* at steady-state as determined by the approximation Mf, preceded by the exact value calculated from the probabilities determined in section 3 above. The approximation is seen to be closer to the true value for small L and for values of r near .5. The result for small L is a result of the assumption that L is much less than M. In Table 3.3, the urn size M is 20, and the values of L near 10 are certainly not small compared than M. Also, as r approaches .5, the expected fraction of red balls in M* approaches ½ as can be seen from Tables 3.1 and 3.2. This coincides with the result (3.8) for r = .5.

	r = .1	Mf	r = .2	Mf	r = .3	Mf	r = .4	Mf	r = .5	Mf
L = 1	2	2	4	4	6	6	8	8	10	10
2	5.14	5	6.76	6.67	7.97	7.92	9.02	8.99	10	10
3	6.75	6.49	7.89	7.73	8.70	8.60	9.37	9.32	10	10
4	7.66	7.32	8.50	8.28	9.08	8.94	9.54	9.49	10	10
5	8.24	7.84	8.88	8.62	9.31	9.15	9.67	9.59	10	10
6	8.64	8.19	9.13	8.85	9.46	9.29	9.75	9.66	10	10
7	8.93	8.44	9.32	9.01	9.57	9.39	9.80	9.71	10	10
8	9.17	8.63	9.47	9.13	9.68	9.47	9.85	9.75	10	10
9	9.36	8.78	9.60	9.23	9.76	9.53	9.89	9.77	10	10
10	9.56	8.90	9.75	9.31	9.86	9.58	9.94	9.80	10	10

Table 3.3. The Expected Number of Red Balls in Urn M* at Steady-State, Followed by the Approximation as Given in (3.8). The Figures are for a System with M = 20.

5. Transfers Between Locations in the Variable List Length Selection Process

In section 3 above, distributional results describing the steady-state composition of urn M* were achieved. Given those results, the movements within the system are of interest.

We first determine the steady-state probabilities that an arbitrary move is a transfer between locations, i.e. is a cost move. We then develop two approximations to the desired probability statements.

Letting $c(n|j;L)$ = Pr {nth move is a cost move given that $R(n-1)=j$ and the selection list is set to be L}, we obtain

$$c(n|j;L) = \frac{\binom{M-j}{L}}{\binom{M}{L}} r + \frac{\binom{j}{L}}{\binom{M}{L}} (1-r).$$

Now letting $c(L)$ denote the steady-state probability that an arbitrary move is a cost move when the selection list has length L, we can remove the condition on $R(n-1)$ obtaining

$$c(L) = \sum_{k=L-1}^{M-L+1} \left(\frac{\binom{M-k}{L}}{\binom{M}{L}} r + \frac{\binom{k}{L}}{\binom{M}{L}} (1-r) \right) \pi(k). \quad (3.9)$$

Once again, we can resort to a simple deterministic approximation. Let $c'(t;L)$ be the approximate cross-transfer rate, i.e. the rate of flow between the two locations. Then, using the arguments leading up to equation (3.7), we have

$$c'(t;L) = \left(\frac{M-\rho(t)}{M}\right)^L r + \left(\frac{\rho(t)}{M}\right)^L (1-r).$$

At steady-state, $\rho = \rho(\infty)$ and $\frac{d\rho}{dt} = 0$. We have

$$c'(L) = \frac{2r(1-r)}{[r^{1/L} + (1-r)^{1/L}]^L} . \quad (3.10)$$

Note that for L = 1, the above equation gives the result obtained in section 3.

The approximation $c'(L)$ can be further approximated. We write $c'(L)$ in the following special way:

$$c'(L) = \frac{2r(1-r)}{2^L \left(1 + \frac{L(r^{1/L}-1) + L(1-r)^{1/L}-1)}{2L}\right)^L} .$$

We now observe the following relations. Let $x = \frac{L}{2}(r^{1/L}-1)$. Then $r^{1/L} = \frac{2x}{L} + 1$. Then $r = (1 + \frac{2x}{L})^L$. Thus as L grows large, r approaches exp(2x) or x approaches $\ell n(r^{\frac{1}{2}})$. Substituting this limiting result into the above equation for c'(L), we obtain

$$c'(L) \cong \frac{r(1-r)}{2^{L-1}\left(1 + \frac{\ell n(r(1-r))^{\frac{1}{2}}}{L}\right)^L} .$$

But the second term in the denominator can also be approximated by an exponential, resulting in the second approximation, denoted c"(L):

$$c''(L) = \frac{(r(1-r))^{\frac{1}{2}}}{2^{L-1}} . \tag{3.11}$$

The last approximation provides some useful insight into the behavior of the system as L is varied. The cross-transfer rate is seen to decrease by a factor of $2^{-(L-1)}$ as L increases. Thus, for L=2 approximately half as many inter-location moves are required as were required for L=1. Maintaining the movement rate for L=1 as a base, we obtain that for L=3 there are approximately ¼ as many inter-location moves, and for L=4, there are approximately 1/8 as many moves.

Thus L is seen to have a profound influence on the cross-transfer rates of the system. Chapter IV develops the difficulties encountered at the personnel distribution center when L is increased. We observe here that the cost savings obtained by decreasing the cross-transfer rate does introduce difficulties in other parts of the system. The magnitude of these difficulties must be weighted against the savings to be expected by increasing L.

Table 3.4 compares the two approximations c'(L) and c"(L) with the exact cross-transfer rate c(L). Examination of the table shows that the approximations are not good for L greater than 6. However, the purpose of the approximations was to gain intuition about the influence of the length of the selection list on the cross-transfer rate. One sees immediately that the approximations under-estimate the influence of L on the transfers. Thus the result above that the number of transfers, or the cross-transfer rate, decreases by $2^{-(L-1)}$ understates the influence of L.

In the next section we generalize the Ehrenfest Decision Model discussed in this section to the case of three locations.

C. THE EHRENFEST DECISION MODEL WITH THREE AND MORE LOCATIONS

1. General Discussion

The generalization of the two location Ehrenfest Decision Model to three and more locations is of interest. The large institutions which are of concern to us throughout

	c(L)	c'(L)	c"(L)
L = 1	.420	.420	.458
2	.213	.219	.229
3	.099	.111	.115
4	.042	.056	.057
5	.017	.028	.029
6	.006	.014	.014
7	.002	.007	.007

Table 3.4. Comparison of Two Approximations to the Cross-Transfer Rate with the Exact Value. The entries are for a system with M=20 and r = .3.

the thesis would normally have personnel stationed at many locations. The effect of various personnel policies on the expected costs to be incurred in the execution of those policies is the subject of investigation.

In this section we are principally concerned with the effect of the length of the selection list on the costs incurred. We begin by looking at the transfers which occur when the system responds to a demand by assigning the first qualified person that it is able to locate. This is the L=1 case. We then introduce L as a parameter in the description of the selection process. We find that in the generalization preferences of various types of transfers must be stated. For a specific preference, we develop the mathematical relations for the one-step transition matrix and then develop steady-state approximations.

We then investigate the costs incurred for the variable L case. We are unable to obtain analytical results in this case, but numerical results can be obtained. We conclude the section with a discussion of the complexities introduced by the generalization. The scope of the added complexities is formulated, giving specific results on the size of the one-step transition matrix and the numbers of calculations which are required to obtain the elements of the matrix.

2. The Three Location Problem with L=1

Suppose now that the institution under study has three locations at which personnel are situated. Except for this modification, the system is assumed to behave in the same manner as that described in section B.2 above.

Here we have arbitrary demands originating from one of three locations, R (red), B (black) and G (green). The demands are processed by the personnel selection center, or manager, again denoted M*. Upon receipt of a demand, M* searches the records of the available personnel in the specified grade until a qualified replacement is found, whereupon that individual is transferred to satisfy the demand. Upon completion of this process, the individual in the position from which the demand originated is made available to M* for reassignment.

To define the composition of the records at M* following the nth transfer, we specify the vector (R(n),B(n),G(n)). The elements of the vector give the number of records at M*

which represent personnel physically located at R, B and G, respectively. Again letting M be the total number of records at M*, i.e. the total number of records of personnel who are available for transfer, we have M = R(n) + B(n) + G(n). Recall that, in the two location problem, specification of R(n) was sufficient to describe the composition of M*. In the three location problem we must specify the current values of any two of the elements of the vector (R(n),B(n),G(n)). Choosing R(n) and B(n), we define M(n) = (R(n),B(n)) as the "composition vector."

Let r, b, and g be the probabilities that an arbitrary demand originates from location R, B and G respectively. Thus we have that r + b + g = 1. A graphical representation of the problem of this section is given in figure 3.2.

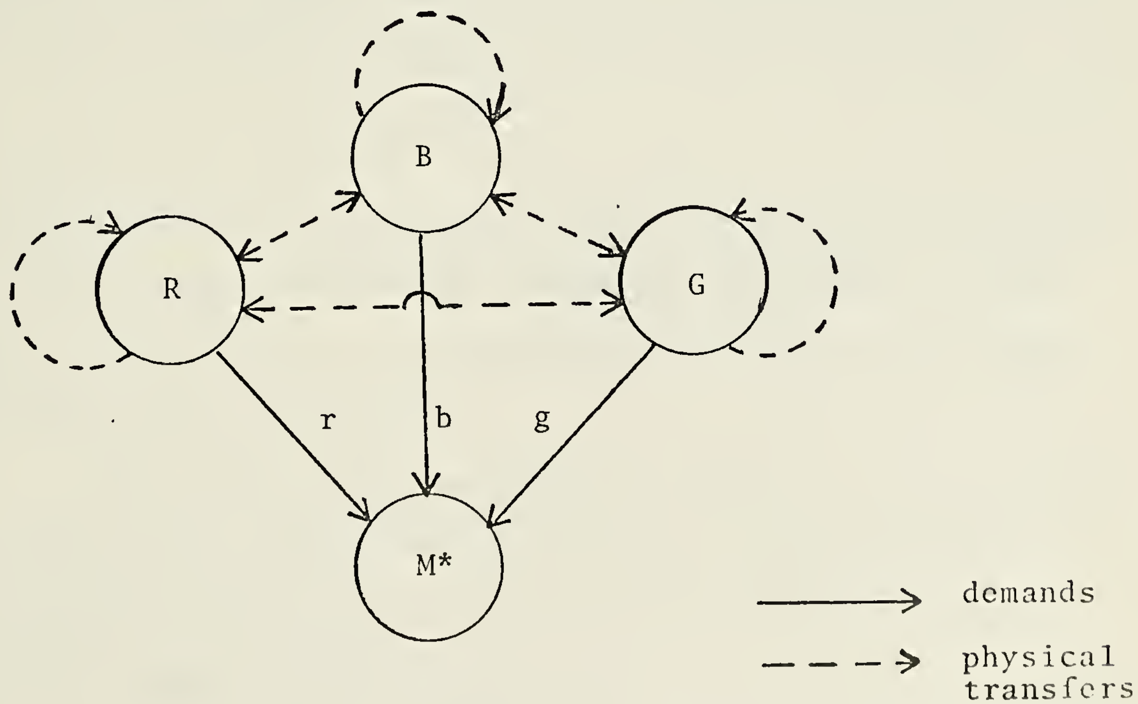


Figure 3.2. A Graphical Representation of the Three Location Ehrenfest Decision Model.

We have the one-step transition probabilities

$$
\begin{aligned}
&\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j,k)\} = \frac{j}{M}r + \frac{k}{M}b + \frac{M-j-k}{M}g, \\
&\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j-1,k+1)\} = \frac{k+1}{M}r, \\
&\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j-1,k)\} = \frac{M-(j-1)-k}{M}r, \\
&\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j+1,(k-1)\} = \frac{j+1}{M}b, \qquad (3.12) \\
&\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j,k-1)\} = \frac{M-j-(k-1)}{M}b, \\
&\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j+1,k)\} = \frac{j+1}{M}g, \\
\text{and}\quad &\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j,k+1)\} = \frac{k+1}{M}g .
\end{aligned}
$$

Graphically, the possible states for urn M*, where M = 5, may be represented as shown in figure 3.3. In this figure, the blackened dots represent the possible feasible states of the system. Mathematically, the feasible states are the non-negative integer solutions to the equation

$$j + k + \ell = M.$$

Note that the j-coordinate represents the number of red balls in urn M*, the k-coordinate represents the number of black balls in urn M* and the ℓ-coordinate represents the number of green balls in the urn.

Note that the planar surface given by $j + k + \ell = M$ is restricted to the positive orthant by virtue of the non-negative requirement, i.e. that there cannot be a negative number of balls in the urn.

Let $p(j,k;n) = \Pr\{M(n)=(j,k)\}$. Then from the system of equations (3.12), we have

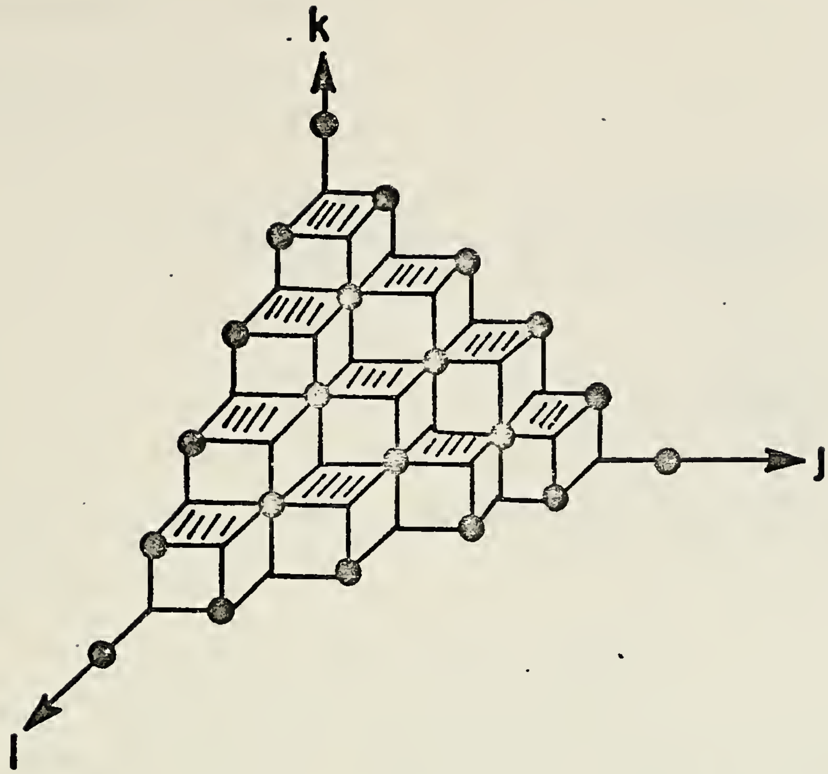


Figure 3.3. Graphical Representation of the Possible States of Urn M*, when M = 5. The blackened dots represent the possible states.

$$p(j,k;n) = \left(\frac{j}{M}r + \frac{k}{M}b + \frac{M-j-k}{M}g\right)p(j,k;n-1) \tag{3.13}$$

$$+ \frac{k+1}{M}rp(j-1,k+1;n-1) + \frac{j+1}{M}bp(j+1,k-1;n-1)$$

$$+ \frac{M-j-k+1}{M}rp(j-1,k;n-1) + \frac{j+1}{M}gp(j+1,k;n-1)$$

$$+ \frac{M-j-k+1}{M}bp(j,k-1;n-1) + \frac{k+1}{M}gp(j,k+1;n-1).$$

Define

$$\theta(z_1,z_2;n) = E\{z_1^{R(n)}z_2^{B(n)}\} = \sum_{j=0}^{M}\sum_{k=0}^{M} z_1^j z_2^k p(j,k;n). \tag{3.14}$$

Substituting (3.13) into (3.14) and assuming steady-state conditions exist, we have

$$\theta(z_1,z_2;\infty) = \theta(z_1,z_2)$$

$$= \frac{rz_1 + bz_2 + g}{(1-(rz_1+bz_2+g))M} \left((1-z_1)\frac{\partial\theta(z_1,z_2)}{\partial z_1} + (1-z_2)\frac{\partial\theta(z_1,z_2)}{\partial z_2} \right) .$$

The solution to this partial differential equation subject to $\theta(1,1) = 1$ is

$$\theta(z_1,z_2) = (rz_1 + bz_2 + g)^M. \tag{3.15}$$

From (3.15) it is immediately recognized that the composition of urn M* obeys a multinomial distribution with parameters M, r and b where $g=1-r-b$.

As in section B.2 above, the next question of concern is that of the probability of an arbitrary move being one between locations. Here we let $c(n|j,k) = \Pr\{\text{inter-location move occurs on the nth transfer given that } M(n-1)=(j,k)\}$. We then obtain

$$c(n|j,k) = 1 - \frac{jr + kb + (M-j-k)g}{M} ,$$

from which, by unconditioning on $M(n-1)$ and assuming steady-state, we obtain

$$c(\infty) = 1 - g^2 - r^2 - b^2.$$

We note that for the two center problem, writing $c(\infty)$ in the above fashion, we obtain $c(\infty) = 1 - r^2 - b^2$, from which we readily obtain $c(\infty) = 2r(1-r)$, which was the result obtained in section B.2.

Comparing (3.12) with (3.1) we note the increase in complexity that results with the addition of another location. The degree of increase in complexity of the mathematical description of the system with increases in numbers of locations is discussed in section 6 below.

As in the case of the two center problem, the next extension is to the case of varying the selection list length. This is the subject of the next section.

3. The Three Location Problem with Variable Length Selection List

In this section, we study the three location Ehrenfest Decision Model under the conditions of a variable length selection list. The length of the list is again denoted L. We see that the analysis becomes much more complicated than the analysis of the two location problem with variable L.

In the multi-location (i.e. greater than two locations) problem, some preferential ordering must be included in the structure of the problem. To examine the need for this ordering, suppose a demand originates at location R. Further suppose that in the selection of candidates to fill the demand, a selection list of length L is required. Upon a search of the available personnel, the list of candidates may include personnel currently located at B and G, but none

located at R. A decision must then be made to send one of the candidates at B or one of the candidates at G.

If the decision is based on the minimization of the transfer cost incurred in satisfying this one demand, then the required preference ordering is easily determined. Suppose that in the institution we are studying the transfer costs between centers have the following relationship:

$$C_{RB} > C_{RG} > C_{BG},$$

where C_{RB} is the cost of transferring one individual from location R to location B. The other costs have similar interpretations. Under the assumption that we are to minimize the cost of each individual transfer, the ordering of preferential transfer may be represented by table 3.5. In table 3.5, lower numbers indicate preferred types of transfers.

		Location of Candidate		
		R	B	G
DEMAND	R	1	4	3
	B	4	1	2
	G	3	2	1

Table 3.5. The Preferential Ordering for Transfers in the Three Location Variable List Length Problem. Lower numbers indicate preferred transfers.

For instance, a demand originating at location R would preferably be met by a candidate currently located at R. Should no candidate from R be included in the L personnel selected, then a candidate from G (if found) would be sent

to R to satisfy the demand. Only in the case that all L selected personnel are currently at B would a person from B be sent to satisfy the demand. Similar interpretations apply to the remaining elements of the matrix.

Graphically, the system may be represented by figure 3.4. The numbers on the dotted arcs indicate the preference ordering on the arcs.

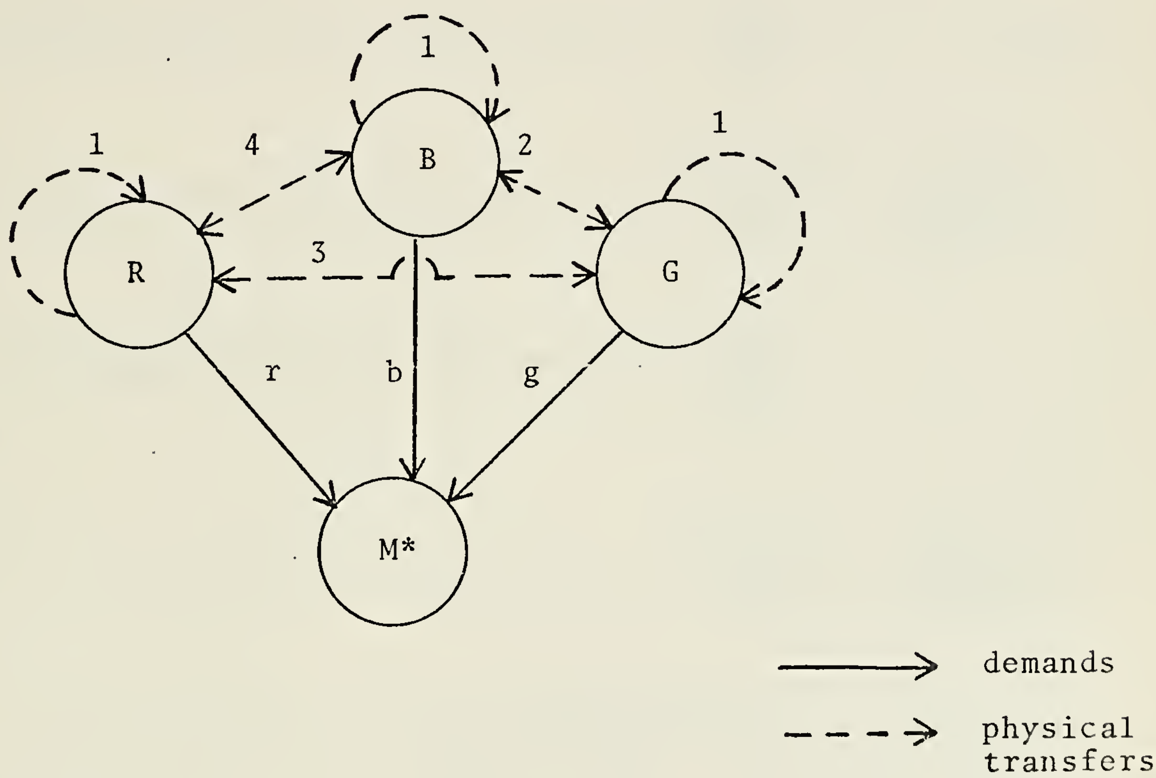


Figure 3.4. A Graphical Representation of the Three Location Ehrenfest Decision Model with Variable Length Selection List. Numbers on arcs represent the preference ordering on transfers.

We can write out the conditional probability statements comparable to (3.3). We have

$$\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j,k)\} = 1 - \frac{\binom{M-j}{L}}{\binom{M}{L}}r - \frac{\binom{M-k}{L}}{\binom{M}{L}}b - \frac{\binom{j+k}{L}}{\binom{M}{L}}g ,$$

$$\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j-1,k+1)\} = \frac{\binom{k+1}{L}}{\binom{M}{L}}r,$$

$$\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j-1,k)\} = \frac{\binom{M-j+1}{L}}{\binom{M}{L}}\left(1-\frac{\binom{k}{L}}{\binom{M-j+1}{L}}\right)r,$$

$$\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j+1,k-1)\} = \frac{\binom{j+1}{L}}{\binom{M}{L}}b, \tag{3.16}$$

$$\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j,k-1)\} = \frac{\binom{M-k+1}{L}}{\binom{M}{L}}\left(1-\frac{\binom{j}{L}}{\binom{M-k+1}{L}}\right)b,$$

$$\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j+1,k)\} = \frac{\binom{j+1}{L}}{\binom{M}{L}}g ,$$

and

$$\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j,k+1)\} = \frac{\binom{j+k+1}{L}}{\binom{M}{L}}\left(1-\frac{\binom{j}{L}}{\binom{j+k+1}{L}}\right)g .$$

To interpret the above equations, let us examine the first, second and third equations above. The first equations simply states that the probability of remaining in the same state is one minus the probability that no reds were selected times the probability that the demand was from red minus other terms with a similar interpretation. This equation describes all type 1 moves, i.e. the most preferred.

The second equation in (3.16) typifies a type 3, or least preferred, move. Here the move described is from B to

R. The equation states that this move occurs only if all candidates selected are currently at B and the demand originated at R.

The third equation in (3.16) is slightly more complicated in nature. It is an example of a type 2 move. The move described is from G to R. This move occurs only when a demand originates at R, there are no reds in the list of candidates and that of the balls selected, not all are black. Note that in the third equation, the statement

$$\frac{\binom{M-j+1}{L}}{\binom{M}{L}}$$

tells us the probability that none of the j-1 red balls in M* were selected. Also the statement

$$1 - \frac{\binom{k}{L}}{\binom{M-j+1}{L}}$$

tells us the probability that not all the balls selected (of which we know no reds to be selected) are black. Thus the denominator in the last term reveals that the population being sampled is only of size $M-(j-1)$, since the population available to sample is known to be non-red.

We are unable to obtain explicit analytical results for the $\pi(j)$'s from $\pi = \pi P$, since inspection shows that from an arbitrary starting point as many as seven states can be reached, one for each equation which applies. However, in a particular application the one-step transition matrix P

can be generated from (3.16). Then the $\pi(j)$'s can be obtained numerically.

In the next section we develop approximations in a manner similar to that used in the two location problem above.

4. Approximations to the Steady-State Probabilities of the Variable List Length Process with Three Locations

We again develop some approximations to the steady-state probabilities given by (3.16). As was done in section B.4, we suppose that L is much smaller than M. This allows us to approximate the above development by using the concept of selection with replacement. Equations (3.16) are then approximated by

$$\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j,k)\} = 1 - \left(\frac{M-j}{M}\right)^L r - \left(\frac{M-k}{M}\right)^L b - \left(\frac{k+j}{M}\right)^L g,$$

$$\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j-1,k+1)\} = \left(\frac{k+1}{M}\right)^L r,$$

$$\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j-1,k)\} = \left(\frac{M-j+1}{M}\right)^L \left(1-\left(\frac{k}{M-j+1}\right)^L\right) r,$$

$$\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j+1,k-1)\} = \left(\frac{j+1}{M}\right)^L b,$$

$$\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j,k-1)\} = \left(\frac{M-k+1}{M}\right)^L \left(1-\left(\frac{j}{M-k+1}\right)^L\right) b,$$

$$\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j+1,k)\} = \left(\frac{j+1}{M}\right)^L g,$$

and $$\Pr\{M(n)=(j,k)\,|\,M(n-1)=(j,k+1)\} = \left(\frac{j+k+1}{M}\right)^L \left(1-\left(\frac{j}{j+k+1}\right)^L\right) g.$$

The above equations are seen to be in powers of L rather than in combinatorials involving L. At this time, we could uncondition on the M(n-1). However, we saw in section B.4 that the result was not very helpful.

We proceed to a deterministic version. Let $\rho(t)$ denote the number of red balls in urn M* at time t. Also let $\beta(t)$ denote the number of black balls in urn M* at time t. Let time be measured in units of transitions. Then

$$\frac{d\rho(t)}{dt} = \left(\frac{\beta(t)}{M}\right)^L r + \left(\frac{M-\rho(t)}{M}\right)^L \left(1 - \left(\frac{\beta(t)}{M-\rho(t)}\right)^L\right) r$$
$$- \left(\frac{\rho(t)}{M}\right)^L (b+g) ,$$

and

$$\frac{d\beta(t)}{dt} = \left(\frac{\rho(t)}{M}\right)^L b + \left(\frac{M-\beta(t)}{M}\right)^L \left(1 - \left(\frac{\rho(t)}{M-\beta(t)}\right)^L\right) b$$
$$\left(\frac{\beta(t)}{M}\right)^L r - \left(\frac{\rho(t)+\beta(t)}{M}\right)^L \left(1 - \left(\frac{\rho(t)}{\rho(t)+\beta(t)}\right)^L\right) g .$$

Letting f(t) denote the fraction of balls in urn M* which are red, i.e. $f(t) = \rho(t)/M$, and h(t) denote the fraction which are black, $h(t) = \beta(t)/M$, we can rewrite the above equations in the fractional form. We obtain

$$M \frac{df(t)}{dt} = (1 - f(t))^L r - f(t)^L (1-r),$$

and

$$M \frac{dh(t)}{dt} = (1 - h(t))^L b - h(t)^L r + f(t)^L g$$
$$- f(t) + h(t))^L g .$$

The latter equations may be obtained using the following argument. We observe from Table 3.3 that location R is the "least preferred" location from which or to which transfers are to be effected. That is, the preference ordering is established such that the higher values are placed on the interaction of R with the other locations.

Thus, transfers into location R occur only if no reds are selected in response to a demand at R. This event occurs with a probability of $(1-f(t))^L r$. Also transfers from R occur only if all the selected balls are red and the demand arose at a location other than red, an event which occurs with probability $f(t)^L(1-r)$.

Similarly transfers into location B occur only if no blacks are selected in response to a demand at B, an event which occurs with probability $(1-h(t))^L b$. Transfers from B are of two types. From Table 3.3 we see that a transfer from B to R is the least preferred of all types of transfers. This occurs in response to a demand at R, and only if all the selected balls are black, i.e. with probability $h(t)^L r$. The other type of transfer from B is in response to a demand at G. If the selections made are all either red or black (i.e., no green balls selected) and there is at least one black ball selected, then a transfer from B to G occurs. This event occurs with probability $(f(t)+h(t))^L g - f(t)^L g$.

At steady-state, the derivatives are zero and the fractions are no longer time-dependent. Letting $f = \lim_{t\to\infty} f(t)$

and $h = \lim_{t\to\infty} h(t)$, we can then write

$$(1-f)^L r - f^L(1-r) = 0$$

and (3.17)

$$(1-h)^L b - h^L r - (f+h)^L g + f^L g = 0.$$

The first equation has the solution given by (3.8). This is expected since the preference ordering of Table 3.3 established location R as the location with the least interaction with the other locations. Thus the other locations could be considered as being one entity and the three location problem reduces to a two location problem insofar as transfers to and from R are concerned.

Unfortunately, we are unable to obtain a closed form solution similar to (3.8) for the second equation in (3.17). In the next section, we investigate the transfer costs incurred in the three location problem.

5. Transfers Between Locations in the Variable Length List Selection Process with Three Locations

In this section, we develop expressions for the probability of incurring a specified cost in response to an arbitrary demand.

Let $C(n;L)$ be the cost of the nth transfer under the policy of a list of length L. From equations (3.16) it is straightforward to show that

$$\Pr\{C(n;L)=0\,|\,M(n-1)=(j,k)\} = 1 - \frac{\binom{M-j}{L}}{\binom{M}{L}} r - \frac{\binom{M-k}{L}}{\binom{M}{L}} b - \frac{\binom{j+k}{L}}{\binom{M}{L}} g,$$

$$\Pr\{C(n;L)=C_{RB}|M(n-1)=(j,k)\} = \frac{\binom{k}{L}}{\binom{M}{L}}r + \frac{\binom{j}{L}}{\binom{M}{L}}b,$$

$$\Pr\{C(n;L)=C_{RG}|M(n-1)=(j,k)\} = \frac{\binom{j}{L}}{\binom{M}{L}}g + \frac{\binom{M-j}{L}}{\binom{M}{L}}r - \frac{\binom{k}{L}}{\binom{M}{L}}r, \tag{3.18}$$

and

$$\Pr\{C(n;L)=C_{BG}|M(n-1)=(j,k)\} = \frac{\binom{M-k}{L}}{\binom{M}{L}}b - \frac{\binom{j}{L}}{\binom{M}{L}}b + \frac{\binom{j+k}{L}}{\binom{M}{L}}g - \frac{\binom{j}{L}}{\binom{M}{L}}g.$$

Once equations (3.16) are solved, the above can be unconditioned and numerical results for the probabilities of incurring the various costs can be obtained.

The above can be approximated in the steady-state case by letting f and h denote the steady-state fraction of red and black balls, respectively, in the urn M*. We then obtain

$$\Pr\{C(\infty;L) = 0\} = 1 - (1-f)^L r - (1-h)^L b - (f+h)^L g,$$

$$\Pr\{C(\infty;L) = C_{RB}\} = h^L r + f^L b,$$

$$\Pr\{C(\infty;L) = C_{RG}\} = f^L g + (1-f)^L r - h^L r, \tag{3.19}$$

and

$$\Pr\{C(\infty;L) = C_{BG}\} = (1-h)^L b - f^L b + (f+h)^L g - f^L g.$$

The solution to equations (3.17), when substituted into the above expressions gives the approximation to the probabilities of incurring a cost in response to an arbitrary demand.

Unfortunately we are unable to obtain analytical results to the above approximations. Thus in the three location problem we are unable to observe directly the change in

the problem with increases in L. The next section discusses some of the obstacles encountered when attempting to generalize to an arbitrary number of locations.

6. The Ehrenfest Decision Model with More Than Three Locations

The preceding sections have shown that the extension of the selection process problem to more than two locations greatly increases the mathematical difficulties encountered. The resulting equations of the model rapidly become less and less tractable.

One, of course, would desire to exhibit some relationships for the cross-transfer rate $c(\infty)$ as a function of the length of the selection list (L), the number of records available to the personnel distribution center (M) and the number of locations in the institution, denoted N. Failing this, however, one might be satisfied with the intuition gained in the two location problem.

For instance, in the two center problem $c(\infty)$ was shown to be approximately proportional to $2^{-(L-1)}$. This rapid decay points to the increase of the selection list L as a means of reducing the costs of inter-location moves. The decision to increase L now seems intuitive in the multi-location case.

Should analytical results be desired in the multi-location case, one first would desire to determine the scope of the problem. We formulate a method to describe the scope of the problem in the multi-location Ehrenfest Decision Model.

In the above sections, two methods of approach have predominated. Backward difference equations have been stated and solved. Also the state dependent Markovian properties have been exploited. General statements can be made about the scope of the analytical problem under both methods.

Lemma 3.4. The number of conditional probability statements required to specify the system with N locations is N^2-N+1.

Proof: There will be one equation giving the conditional probability of remaining in the same state. For each of the locations considered to be a source, there are N-1 locations which are possible destinations. The conditional probability of a transfer from one location to each other is the basis for one of the backward equations.

Hence $N(N-1) + 1$ equations are required to specify the system.

Note however, that the specification of the system equations is just a starting point under this method. The condition has to be removed from the conditional probability statements and the result interpreted. In the above sections, we found the probability generating functions to be useful in the interpretation. It was through the probability generating functions that we were able to see that the composition of urn M^* obeys a binomial distribution in the two location case and a "trinomial" in the three location case.

Now we consider the state dependent Markovian approach to the description of the system. Let $P(N;M)$ be the matrix of one-step transition probabilities. It is known that $P(N;M)$ is a square matrix with the dimension being the number of possible states in the system. Let $|P(N;M)|$ denote the dimension of the matrix $P(N;M)$.

Lemma 3.5. The dimension of $P(N;M)$, i.e. $|P(N;M)|$, is given by $\binom{M+N-1}{N-1}$.

Proof. (By induction)

It was shown in the two center problem that $|P(2;M)| = M + 1$. Also, $\binom{M+2-1}{2-1} = M + 1$. Hence the lemma is true for $N = 2$.

Suppose the lemma is true for $N=n$. We show this implies the lemma is true for $N=n+1$. First suppose that the $n+1$st location is excluded from consideration. Then the dimension of the one step transition matrix is $|P(n;M)|$. Suppose now that exactly one person from the $n+1$st location is in the urn M^*. Then the remaining $M-1$ personnel in urn M^* are distributed among the first n locations, thus giving rise to a one step transition matrix with dimension $|P(n;M-1)|$. This process continues until there are M personnel from the $n+1$st location in the urn M^* and hence none from any other location. Thus there are

$$|P(n;M)| + |P(n;M-1)| + \cdots + |P(n;0)|$$

possible states in a system with $n+1$ locations and urn size M. Thus

$$P(n+1;M) = \binom{M+n-1}{n-1} + \binom{M-1+n-1}{n-1} + \cdots + \binom{n-1}{n-1} .$$

Observing from the Pascal triangle that

$$\binom{k+n}{n} = \binom{k+n-1}{n-1} + \binom{k+n-1}{n}$$

$$= \sum_{j=0}^{k} \binom{j+n-1}{n-1}$$

it is readily seen that

$$|P(n+1;M)| = \binom{M+n}{n} .$$

Hence we conclude by induction that $|P(N;M)| = \binom{M+N-1}{N-1}$ for $N \geq 2$.

The elements of the matrix $P(N;M)$ are given by at most $|P(N;M)|^2$ probability statements. Consider, however, the nature of the allowed transitions. Due to the restriction that only two elements of the composition vector can change in response to any given demand, that the change incurred can be no larger than one unit, and that an increase in one element must be matched by a decrease in another element, considerably fewer probability statements need be determined than the maximum stated above.

Once determined, the matrix $P(N;M)$ raised to successive integral powers gives matrices approaching a matrix, the rows of which are the limiting steady-state probabilities. Thus if $|P(N;M)|$ is not prohibitive, results can be determined. Also, $\pi = \pi P$ can be solved for the limiting steady-state probabilities.

Thus we have been able to at least quantify the scope of the problem for the multi-location Ehrenfest Decision Model. The latter approach, i.e. using the state dependent Markovian properties, seems to be more conducive to the obtaining of results than the probability generating function approach. Although the initial statement of the problem may be tedious, once stated the problem may be solved.

The length of the selection list enters the problem in the determination of the elements of P(N;M). Thus as L is varied, the probability statements forming the elements of P(N;M) must be changed. Since this is the portion of the problem which may be tedious, approximations may be useful at this stage. As in section C.4, it may prove helpful to formulate the problem as one with replacement. This leads to statements that are in terms of the Lth power of a fraction, rather than combinatorial terms in L. Thus the approximations may be more readily calculated than the exact expressions for varying L.

D. DIFFUSION APPROXIMATIONS

1. General Discussion

Diffusion processes are Markov processes in which only continuous changes of states occur (see Cox and Miller [7], page 203). For a more mathematical description, see Feller [10]. Diffusion processes are often derived by the scheme of dividing the time axis and the displacement axis (for a one-dimensional process) into a very large number of

small intervals. A discrete Markov process, such as a random walk, is then observed to occur on the fine grid thus established. In many cases the limit of the discrete Markov process as the increments on the time axis and displacement axis are forced to become smaller and smaller is a diffusion process.

More formally, consider a stochastic process $\{X(t),\ t \geq 0\}$. Let $u_o(X(t))$ be a continuous, bounded function of $X(t)$. Further, let $u(t,x)$ be the expectation of $u_o(X(t))$ conditional on the hypothesis that $X(o) = x$. If $u(t,x)$ satisfies the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{a}{2}\ \frac{\partial^2 u}{\partial x^2} + b\ \frac{\partial u}{\partial x}\ ,$$

where

$$u(t,x) = \int u_o(y) q_t(x,y)dy,$$

$q_t(x,y)$ being the density function for the transition probability, i.e. for the probability that the random function $X(t)$ will be in the interval $(y,y+dy)$ at time t given that it had the value x at time 0, and

(1) For fixed $\Delta>0$, $\int_{|y-x|\geq\Delta} q_t(x,y)dy$ is $o(t)$,

(2) $\frac{1}{t} \int_{|y-x|<\Delta} (y-x)q_t(x,y)dy \to b(x)$ as $t\to 0$,

and

(3) $\frac{1}{t} \int_{|y-x|<\Delta} (y-x)^2 q_t(x,y)dy \to a(x)$ as $t\to 0$,

then u is said to be a diffusion process. The limiting quantity b(x) is often called the infinitesimal velocity or drift. The quantity a(x) is the infinitesimal variance. For more detail and examples, see Feller [10].

In other cases, a discrete Markov process may be approximated by a diffusion process. This approach is essentially the reverse of the limiting scheme discussed above. The procedure is useful because of the resultant gain in mathematical tractability of the problem when the continuum is used for the sample space. In many cases the use of diffusion approximations to discrete Markov processes has enabled the constructors of stochastic models to describe time-dependent phenomena that otherwise defied description or, at best, obscured the insight provided by description through the use of more conventional methods.

For instance, Gaver [12] applies the use of a diffusion approximation to the description of the waiting time in the M/G/1 queue. In the introduction to that paper, he states "the understanding of system performance furnished by the present mathematical theory is inadequate. The reason is that while the consideration of simple problems typically yields elegant mathematical results, the form of these results...is not immediately comprehensible nor useful in simple comparisons."

Gaver and others have found diffusion approximations useful in many other cases [13,14]. Many other people have

used the diffusion approximation, including Newell [31,32], Kingman [25], Kimura [24] and Iglehart [22].

There are two basic approaches to the application of the diffusion approximation to a particular discrete Markov process. One, used by Gaver, Lehoczky, and Perka [15], goes directly to stochastic differential equations which describe the movements within the system. The solution to that equation then leads to the approximate distribution required. Another, given in Cox and Miller [7], involves an examination of the fundamentals of the limit-taking procedure. The limits must be taken in such a manner as to not obscure the randomness that is being described. That is, if the increments on the displacement axis decrease as fast as or faster than the increments on the time axis, the resulting limits give a deterministic system. Also, as the increments on the displacement axis become smaller, the number of increments must increase in such a manner that the product of the number of increments and the size of the increments increases.

Both of the above approaches have the same end result, so they appear equivalent. We choose the latter approach, since the limit-taking process provides insight into the application.

2. <u>The Two Location Problem with L=1</u>

This section develops the diffusion approximation to the model of section B.2 above. In order to proceed, it is necessary to redefine the system parameters. Although

in the previous development it was found to be sufficient to specify only the number of reds in the urn M*, here we use a function of both the number of red balls and the number of black balls in the urn.

Consider the process $Y(n)=R(n)-B(n)$, where $R(n)+B(n)=M$, the total number of balls in the urn M*. The process $Y(n)$ is characterized by

$$Y(n+1) = Y(n) + Z(n+1), \qquad (3.20)$$

where $Z(n+1)$ is a random variable which is dependent on $Y(n)$. The distribution of $Z(n+1)$ is developed shortly. In keeping with the philosophy of the diffusion approximation, the magnitude of the step is to be variable. We assume there are $M\Delta$ balls in the urn. If $R(n)$ increases by Δ, then $B(n)$ decreases by Δ, and the change in $Y(n)$ is seen to be 2Δ. Similarly, if $R(n)$ decreases by Δ, then $B(n)$ increases by Δ, and the change in $Y(n)$ is -2Δ. We then have that $R(n) + B(n) = M\Delta$. Then $M\Delta+Y(n) = 2R(n)$ and $M\Delta-Y(n) = 2B(n)$.

Adding $M\Delta$ to both sides of equation (3.20), using the above relations and dividing by 2, we obtain

$$R(n+1) = R(n) + \tfrac{1}{2}Z(n+1).$$

This is a characterization of the process described in section B.2, if $\Delta=1$. Thus from (3.1) we can write directly the distribution of $Z(n+1)$:

$$\Pr\{Z(n+1)=2\Delta \mid Y(n)\} = \left(1 - \frac{Y(n)}{M\Delta}\right)\frac{r}{2}, \qquad (3.21)$$

$$\Pr\{Z(n+1)=0 \mid Y(n)\} = \left(1 + \frac{Y(n)}{M\Delta}\right)\frac{r}{2} + \left(1 - \frac{Y(n)}{M\Delta}\right)\frac{1-r}{2},$$

and

$$\Pr\{Z(n+1)=-2\Delta \mid Y(n)\} = \left(1 + \frac{Y(n)}{M\Delta}\right)\frac{1-r}{2}.$$

Now the expected value of $Y(n+1)$ is determined. Taking the expected value of both sides of (3.20), conditioned on knowledge of $Y(n)$, we obtain

$$E\{Y(n+1) \mid Y(n)\} = \Delta(2r-1) + \left(1 - \frac{1}{M}\right) Y(n).$$

Letting $p=2r-1$ and $a=1-\frac{1}{M}$, we have $E\{Y(n+1) \mid Y(n)\} = \Delta p + aY(n)$. Given an initial condition $Y(0) = R(0)-B(0) = y_o$, we have

$$E(Y(1)) = \mu(1) = \Delta p + ay_o.$$

<u>Lemma 3.6.</u> In general

$$E\{Y(n)\} = \mu(n) = \Delta p(1+a+a^2+\ldots+a^{n-1}) + a^n y_o. \qquad (3.22)$$

Proof. The lemma was shown to hold for $\mu(1)$ in the above discussion. The proof is then immediate by induction.

Now for $\Delta=1$, the limit as n grows large of $E\{Y(n)\} = \frac{p}{1-a} = Mp = M(2r-1)$. But $Y(n) = R(n) - B(n) = 2R(n) - M$. Also from section B.2, we have $ER(\infty) = Mr$. Hence, $EY(n) = 2Mr-M = M(2r-1)$. Thus, the process $Y(n)$ is at least consistent with the earlier work in expectation.

An expression for the variance of $Y(n)$, denoted $V\{Y(n)\}$ or $\sigma^2(n)$, is now required. We obtain

$$V\{Y(n+1) \mid Y(n)\} = \Delta^2(1 + 4r(1-r)) + Y(n)^2/M.$$

When the condition on Y(n) is removed, we obtain

$$\sigma^2(n+1) = (1 - \frac{2}{M})\, \sigma^2(n) + 8\Delta^2 r(1-r)$$
$$- a^n 2\Delta p \left(\frac{y_o}{M} - \Delta p \right) - a^{2n} \left(\frac{y_o}{M} - \Delta p\right)^2. \tag{3.23}$$

For the initial condition $Y(0) = y_o$, we have $\sigma^2(0) = 0$. Let $b = 1 - \frac{2}{M}$, $c = 8\Delta^2 r(1-r)$ and $d = \frac{y_o}{M} - \Delta p$. Then (3.23) becomes

$$\sigma^2(n+1) = b\sigma^2(n) + c - a^n 2\Delta pd - a^{2n} d^2.$$

For n=0, this becomes

$$\sigma^2(1) = c - 2\Delta pd - d^2.$$

<u>Lemma 3.7.</u> In general,

$$\sigma^2(n) = \frac{1-b^n}{1-b} c - \frac{a^n - b^n}{a-b} 2\Delta pd - \frac{a^{2n} - b^n}{a^2 - b} d^2. \tag{3.24}$$

Proof. Again, the proof follows immediately by induction. Again, for $\Delta=1$, $\sigma^2(\infty) = \frac{c}{1-b} = \frac{8r(1-r)}{1-(1-2/M)} = 4Mr(1-r)$. From section B.2, we obtained $V\{R(n)\} = Mr(1-r)$. Hence $V\{Y(n)\} = V\{2R(n)-M\} = 4 \mathrm{Var} R(n) = 4Mr(1-r)$ is again consistent.

Lemmas 3.6 and 3.7 give expressions in the discrete time, indexed by n. We need to change from this time scale to a continuous time scale. We do this by setting $t = n\tau$ and letting τ, the time interval between transactions, approach zero.

For the diffusion approximation to be applicable, we need $\mu(t)$, the continuous analog of $\mu(n)$, to approach a constant. We also need the variance to approach a constant, i.e. $\sigma^2(n)$ approaches some constant k. Note that if k were zero, the system would become deterministic. To avoid the possibility of the system becoming deterministic, it is necessary that the step size Δ decrease at a slower rate than τ. The choice $\Delta = \tau^{\frac{1}{2}}$ is appropriate.

The relationship among the parameters of the problem are given:

$$
\begin{aligned}
r &= \tfrac{1}{2} + \gamma\tau^{\frac{1}{2}}, \\
\Delta &= \tau^{\frac{1}{2}}, \\
M\Delta^2 &= k,
\end{aligned}
\tag{3.25}
$$

and

$$t = n\tau,$$

where γ and k are constants.

Substituting into equation (3.22), we obtain

$$\mu(t) = 2\gamma k + \left(1 - \frac{\tau}{k}\right)^{t/\tau} (y_o - 2\gamma k)$$

which in the limit as τ approaches zero becomes

$$\mu(t) = 2\gamma k + (y_o - 2\gamma k)\exp(-t/k). \tag{3.26}$$

Similarly, substituting into (3.24) and allowing τ to approach zero, we obtain

$$\sigma^2(t) = k\ (\ 1 - \exp(-2t/k)\ ). \tag{3.27}$$

In the limit as t becomes large, equations (3.26) and (3.27) assume the steady-state value. We obtain

$$\ddot{\mu} = \lim_{t\to\infty} \ddot{\mu}(t) = 2\gamma k$$

and

$$\bar{\sigma}^2 = \lim_{t\to\infty} \sigma^2(t) = k.$$

Since k and γ were arbitrary constants, the last relation determines k as the steady-state variance and thus γ is seen to be $\mu/2\sigma^2$. Now (3.26) and (3.27) may be re-written as

$$\mu(t) = \mu + (y_0 - \mu)\exp(-t/\sigma^2) \tag{3.28}$$

and

$$\sigma^2(t) = \sigma^2\ (1 - \exp(-2t/\sigma^2)). \tag{3.29}$$

Having obtained expressions for $\mu(t)$ and $\sigma^2(t)$, we now develop the distribution function which Y(t) obeys. We can state that

$$Y(t+\tau) = Y(t) + Z(t+\tau),$$

where the distribution of $Z(t+\tau)$ is obtained from equations (3.21) by direct substitution. Looking then at the moment generating function conditioned on Y(t),

$$\theta(t+\tau;m) = E\{\exp(mY(t+\tau))\} = E\ E\{\exp(mY(t+\tau))|Y(t)\}.$$

Letting $A = e^{m\tau^{\frac{1}{2}}} - e^{-m\tau^{\frac{1}{2}}}$ and $B = e^{m\tau^{\frac{1}{2}}} + e^{-m\tau^{\frac{1}{2}}}$, we obtain

$$\theta(t+\tau;m) = \theta(t;m) + \tfrac{1}{4}\theta(t;m)A(A+2\gamma\tau^{\frac{1}{2}}B)$$
$$- \frac{1}{4M\tau^{\frac{1}{2}}}\ \frac{\partial\theta(t;m)}{\partial m}\ A(B+2\gamma\tau^{\frac{1}{2}}A),$$

which in the limit as τ approaches zero becomes

$$\frac{\partial\theta(t;m)}{\partial t} = (m^2 + 2\gamma m)\theta(t;m) - \frac{m}{\sigma^2}\frac{\partial\theta(t;m)}{\partial m} .$$

We now have a partial differential equation in two independent variables. Substituting $\gamma = \mu/2\sigma^2$ into the above equation gives

$$\frac{\partial\theta(t;m)}{\partial t} = \frac{m}{\sigma^2}\{(\mu + m\sigma^2)\theta(t;m) - \frac{\partial\theta(t;m)}{\partial m}\}. \qquad (3.30)$$

To solve this partial differential equation, we follow the procedure used by Cox and Miller [7]. Let $K(t;m)$ be the cumulant generating function, i.e.

$$K(t;m) = \ell n\ \theta(t;m).$$

Then equation (3.29) becomes

$$\frac{\partial K(t;m)}{\partial t} + \frac{m}{\sigma^2}\frac{\partial K(t;m)}{\partial m} = m\sigma^{-2}\ (\mu + m\sigma^2). \qquad (3.31)$$

With the initial condition $Y(0) = y_o$, equation (3.30) has the solution

$$K(t;m) = m\mu + m(y_o - \mu)\exp(-t/\sigma^2) + \frac{m^2\sigma^2}{2}(1 - \exp(-2t/\sigma^2)),$$

or (3.29) has the solution $\exp(K(t;m))$, from which $\theta(t;m)$ is readily recognized as the generating function for the normal distribution. Furthermore, using equations (3.25) and (3.26), we have

$$\theta(t;m) = \exp(m\mu(t) + \frac{m^2}{2}\sigma^2(t)). \qquad (3.32)$$

Hence $Y(t)$ is seen to be a time varying normal process with mean $\mu(t)$ and variance $\sigma^2(t)$. Further, as time increases the process settles down to a stationary normal process with parameters μ and σ^2. That is,

$$\lim_{t\to\infty} \theta(t;m) = \theta(\infty;m) = \exp(m\mu + \tfrac{1}{2}m^2\sigma^2).$$

Thus we have shown that the selection process for the two location institution can be approximated by a diffusion process. From this approximation, one is able to obtain insight into the nature of the transient process, i.e. how the composition of urn M* behaves following some initial disturbance away from the steady state case.

In the next section, we develop the diffusion approximation for the two location, variable length selection list problem.

3. The Two Location Problem with Variable Length Selection List

In this section, we extend the results of the previous section to the case of the variable length selection list. Again, we examine the process $Y(n) = R(n) - B(n)$, where the process is characterized by equation (3.20).

In order to define the conditional distribution of $Z(n+1)$ given $Y(n)$, it is convenient to use the approximation of section B.4. Using equation (3.3a) from that section, the analog of equations (3.21) becomes

$$\Pr\{Z(n+1)=2\Delta|Y(n)\} = 2^{-L}(1-\frac{Y(n)}{M\Delta})^{L} r \quad (3.33)$$

$$\Pr\{Z(n+1)=0\ |\ Y(n)\} = 1-2^{-L}\{(1-\frac{Y(n)}{M\Delta})^{L} r+(1+\frac{Y(n)}{M\Delta})^{L}(1-r)\},$$

and

$$\Pr\{Z(n+1)=-2\Delta|Y(n)\} = 2^{-L}(1+\frac{Y(n)}{M\Delta})^{L}(1-r).$$

Making use of the relations (3.25), the development is quite similar to that used in the previous section. Let $\mu(t;L)$ denote the expected value of $Y(t)$ under the policy of a selection list of length L. Also let $\sigma^2(t;L)$ denote the variance of $Y(t)$ under the policy of a list of length L. It is convenient in this case to amke the arbitrary constant γ be a function of L, denoted $\gamma(L)$ where $\gamma(L) = \gamma(1)L$, $\gamma(1)$ being the γ of the previous section. Then

$$\mu(t;L) = \mu + (y_0-\mu)\exp(-\frac{2Lt}{2^L\sigma^2}) \quad (3.34)$$

and

$$\sigma^2(t;L) = \frac{\sigma^2}{L}(1-\exp(-\frac{4Lt}{2^L\sigma^2})), \quad (3.35)$$

where μ and σ^2 are the limiting results obtained in the preceeding section.

The generating function is now also indexed on the length of the list L, and is denoted $\theta(t;m;L)$. It is relatively straightforward to show that

$$\theta(t;m;L) = \exp(m\mu(t;L) + \frac{m^2\sigma^2}{2}(t;L)),$$

which is again recognized to be that of a normal process. We note that as L increases, the variance of the approximation

goes down by L^{-1} as steady state is approached. Also, the exponential decay rate decreases by a factor of $L/2^{L-1}$. Thus, there will be less variance once steady state is achieved, but it will take longer to attain steady-state conditions. This is intuitive, since the selection process described by the variable L case is tending to maintain the status quo, i.e., is designed to reduce the flows between locations. This in turn prevents the system from rapidly attaining steady state.

E. EXTENSIONS AND CONCLUSIONS

In this chapter, we have developed a model of the personnel selection process within a large institution. The selection process was seen to be first concerned with finding a suitable (qualified) individual in response to demand for a person with specified qualifications. Of secondary concern was the current location of the selectees.

As the length of the selection list, L, was allowed to grow, we saw the steady-state inter-location transfer cost decrease. Thus, if the institution is concerned about decreasing the transfer costs then the personnel managers should be tasked with finding more qualified personnel prior to effecting a transfer, under the assumptions of the chapter.

The model was developed for an institution with two locations, then extended to the case of an institution with three locations. Comments about extensions to institutions with more locations were made.

Approximations were developed for steady state results. Also diffusion approximations were developed to aid in describing the transient flow given some deviation from steady-state as an initial condition.

Throughout the chapter, we were implicitly concerned with the minimization of the transfer cost in response to individual demands. Such a myopic view of the selection process does not consider the long range needs of the institution, nor does it consider that moves effected early in the period can influence later moves. That is, of the L selectees the one associated with the least transfer cost may be more suitably held in the urn M* for a later demand. In that case, it might be desired to transfer an individual with a slightly higher cost of relocation.

The results of chapter II are seen to impinge on the personnel selection process at this point. The use of the model as a personnel planning model gave as an output the optimal set of inter-location transfers by pay grade. If pay grade is the determining factor in the qualification specified in this chapter, then the institution would desire to effect a transfer only if that particular transfer were in the optimal set provided by the periodic planning model. Of course, this may not be possible, since it is not guaranteed that such a transfer is included in the options upon completion of a particular selection of L candidates. Such an event would imply a higher cost of transfers than was provided by the planning model. However, the

use of such a scheme would tend to avoid the gross suboptimization of the selection process without such considerations.

In the next chapter, we investigate the problems encountered by the personnel distribution center when it is forced to select more and more candidates prior to effecting a transfer, reflecting an increase in L.

IV. THE RECORD SEARCH PROBLEM

A. GENERAL DISCUSSION

In chapter III, we saw how the expected cost of transfers is reduced by increasing the length of the selection list. In this chapter, we develop a method of determining the time required for the personnel distribution center, M*, to establish a selection list of length L. Obviously, if the time required to establish each individual list is too great, the personnel distribution center could encounter operating difficulties.

Let us approach the problem from the standpoint of the personnel distribution center. Suppose there are a fixed number of records, representing personnel available for transfer, at the disposal of the distribution center. Let the number of records be denoted N. These records may be considered to be the personnel files for individuals physically located at many different locations. From the N records, the distribution center M* must either

(1) select an individual to fill a vacancy (L=1),

(2) select a group of individuals as candidates for the vacancy (variable L, L greater than 1), or

(3) demonstrate that there is no qualified replacement among the N available records.

The procedure to be followed will be an investigation of the problem when the number of records is very large. Restrictions placed on the system by increasing the length

of the list L and by restricting the number of records available are then investigated.

B. THE BASIC PROBLEM -- UNRESTRICTED NUMBER OF RECORDS

1. General Distributional Results

Suppose that each record made available to the distribution center M* has a probability $m(j)$ of being that of a person qualified for a job of category j. For brevity, we will say that this person is of category j. Suppose $m(j)$ is independent of all factors except the category itself.

Let $M(j)$ be the number of records among the N available which represent personnel of category j. Under the independence assumption above, $M(j)$ is distributed binomially with parameters N and $m(j)$, i.e.

$$\Pr\{M(j)=n\} = \binom{N}{n} m(j)^n (1-m(j))^{N-n}.$$

Let $F(j;k)$ be the number of records of personnel <u>not</u> of category j searched prior to finding the kth person of category j. We assume that the records are observed in a random fashion, i.e. that no sorting has been done or any other procedure which would tend to make the qualifications of one record dependent on the records searched in the past. We ignore momentarily any restriction on the number of records available. That is, if N is large enough, we will always be able to find the k personnel having the specified qualification. Then, $F(j;k)$ is seen to have a negative binomial distribution with parameters k and $m(j)$. That is,

$$\Pr\{F(j;k)=n\} = \binom{n+k-1}{k-1} m(j)^k (1-m(j))^n .$$

Suppose now that we restrict ourselves to searching for personnel of category j. We then can suppress the j, letting $m(j) = m$.

Given $F(j;k)$, the total number of records searched to find the kth person of category j is then $F(j;k) + k = G(j;k)$. Let us define the generating function of $G(j;k)$ as

$$\hat{G}(j;k;z) = E\{z^{G(j;k)}\}.$$

Then

$$\hat{G}(j;k;z) = \left(\frac{zm}{1-z(1-m)}\right)^k . \qquad (4.1)$$

Now the expected value of $G(j;k)$ is given by the first derivative of (4.1) with respect to z, the result evaluated at z=1. This gives us

$$E\{G(j;k)\} = k/m .$$

Also, the variance of $G(j;k)$ is given by

$$\left.\frac{d^2\hat{G}(j;k;z)}{dz}\right|_{z=1} + E\{G(j;k)\} - E^2\{G(j;k)\},$$

from which

$$V\{G(j;k)\} = \frac{k(1-m)}{m^2} .$$

We thus have obtained an expression for the generating function of the number of records searched and have determined the mean and variance of the distribution.

Again, leaving N, the number of available records, unrestricted, we introduce the second attribute of the selected candidates. Here the second attribute of concern is current physical location. Let $\ell(j;i)$ be the probability that an arbitrarily selected person of category j is currently located at i. Suppose $\ell(j;i)$ to be a function of only the category j and the location i. Let $Q(j;i)$ be the number of qualified personnel of category j selected in order to obtain the first person of category j located at i.

$$\Pr\{Q(j,i)=k\} = (1-\ell(j,i))^{k-1}\ell(j,i), \quad k=1,\ldots$$

Also

$$E\{z^{Q(j;i)}\} = \frac{z\ell(j,i)}{1-z\{1-\ell(j,i)\}},$$

or $Q(j,i)$ is seen to be distributed geometrically with parameter $\ell(j,i)$.

Now let $R(j,i)$ be the number of records searched to find the first qualified person of category j at location i. Then

$$E\{z^{R(j,i)}|Q(j,i)=k\} = E\{z^{G(j,k)}\},$$

which is given by (4.1). Thus, removing the condition on $Q(j,i)$,

$$E\{z^{R(j,i)}\} = \frac{zm(j)\ell(j,i)}{1-z\{1-m(j)\ell(j,i)\}}, \tag{4.2}$$

from which $R(j,i)$ is seen to be distributed geometrically with parameter $m(j)\ell(j,i)$.

In the next section, we discuss the distribution of the time required to search through the R(j,i) records to find the first qualified person at the specified location.

2. Time Requirements for Basic Record Search Problem

Given the number of records required to be searched to find the first person of category j at location i, we are interested in determining the distribution of the time expended in the search.

Suppose the time to search the kth record is denoted X(k,t). Suppose all searches of records obey the same distribution and that furthermore, the time to search any record is independent of the time to search all other records. That is, the times are independent and identically distributed (i.i.d.) random variables.

Let T(j,i) be the time required to find the first person of category j at location i. Then, for a given R(j,i),

$$T(j,i) = X(1,t) + \cdots + X(R(j,i),t),$$

since R(j,i) is the number of records which must be searched. Since the random variables are i.i.d. the k can be suppressed, and the random variable is now denoted X(t). Now let

$$X(s) = E\{ e^{-sX(t)} \}.$$

Then

$$E\{ e^{-sT(j,i)} \mid R(j,i)=n \} = X(s)^n,$$

from which, by removing the condition on R(j,i) we obtain

$$\hat{T}(j,i;s) = E\{ e^{-sT(j,i)} \}$$

$$= \frac{X(s)m(j)\ell(j,i)}{1 - X(s)\{1-m(j)\ell(j,i)\}}, \tag{4.3}$$

which is easily shown by substitution into (4.2). From (4.3), it is obtained that

$$ET(j,i) = ER(j,i)EX(t) = \frac{EX(t)}{m\ell}$$

and

$$\mathrm{Var}T(j,i) = ER(j,i)\mathrm{Var}X(t) + E^2X(t)\mathrm{Var}R(j,i)$$

$$= \frac{\mathrm{Var}X(t)}{m\ell} + E^2X(t) \frac{1-m\ell}{(m\ell)^2}.$$

In the special case that $X(t)$ is the exponential distribution with mean μ^{-1}, $T(j,i)$ can be shown to obey the exponential distribution with mean $\{\mu m(j)\ell(j,i)\}^{-1}$.

Thus under assumption of independent, identically distributed times to search individual records, of an unlimited number of records available, and of no restrictions on the number of qualified personnel found prior to effecting a transfer, we have an expression for the transform of the distribution of the time to find the first person of category j who is currently located at i. If these assumptions held, the system could be operated with no cross-transfers and hence no inter-location moving costs, provided only that the time to sort through the possibly large number of records is immaterial.

We next look more closely at the reason for searching to find a number of qualified personnel prior to making

a decision as to which person to transfer in response to a given demand.

C. RESTRICTIONS ON THE LENGTH OF THE SELECTION LIST

The requirement for the personnel distribution center to search until finding L qualified candidates as was required in chapter III, is somewhat arbitrary. In this section, we investigate an alternative formulation of the problem.

Consider the two location problem of section III.B.3 where the variable length selection list problem was introduced. There, after selecting L candidates in response to a demand, the current locations of the candidates were examined to determine whether or not any one of the candidates were at the location from which the demand originated. Another policy might be to examine the current location of each selected candidate to determine if he is at the location from which the demand originated. The search continues until either the first person at the proper location is selected or L qualified personnel have been selected, none of whom are currently at the location from which the demand originated. In this case, L is an upper limit on the number of qualified replacements to be found prior to effecting a transfer. As in the previous case, L is a control variable, i.e. subject to institutional policy.

Suppose now that the length of the selection list is restricted to a maximum value, denoted L. Let $R(j,i;L)$ be

the number of records searched to find the first person of category j at location i or L people of category j, whichever occurs first.

Since the attributes of category (or qualification) and location of the individual represented by any particular record are assumed independent of any previous records searched, a renewal theory argument tells us

$$\begin{aligned} R(j,i;L) &= 1 && \text{with probability } m\ell, \\ &= 1 + R(j,i;L-1) && \text{with probability } m(1-\ell), \\ \text{and} \qquad &= 1 + R'(j,i;L) && \text{with probability } 1-m, \end{aligned}$$

where $m = m(j)$ is the probability that an arbitrarily selected record is of category j, $\ell = \ell(j;i)$ is the probability that an arbitrarily selected person of category j is at location i, and $R'(j;i;L)$ is a random variable with the same distribution as $R(j,i;L)$. The relations hold for L greater than one. For L=1, we have

$$\begin{aligned} R(j,i;1) &= 1 && \text{with probability } m \\ \text{and} \qquad &= 1 + R'(j,i;1) && \text{with probability } 1-m, \end{aligned}$$

since for L=1, the search terminates upon selection of the first qualified person.

Hence the generating function $\hat{R}(z;1) = E\{z^{R(j,i;1)}\}$ is given by

$$\hat{R}(z;1) = \frac{zm}{1 - z(1-m)} . \qquad (4.4)$$

We can also easily obtain the generating function $\hat{R}(z;L) = E\{z^{R(j,i;L)}\}$ in terms of $\hat{R}(z;L-1)$. The result is

$$\hat{R}(z;L) = \frac{zm\ell + zm(1-\ell)\hat{R}(z;L-1)}{1 - z(1-m)} . \tag{4.5}$$

Defining $\hat{R}(z;0) = 1$, the above reduces to (4.4).

Lemma 4.1. The generating function for the process with a selection list of length L is given by

$$\hat{R}(z;L) = \frac{1}{1-z(1-m\ell)} \left(zm\ell + (1-z)\left(\frac{zm(1-\ell)}{1-z(1-m)}\right)^{L}\right). \tag{4.6}$$

Proof. The proof is immediate by induction.

From (4.6), we obtain the limiting result,

$$\hat{R}(z;\infty) = \lim_{L\to\infty} \hat{R}(z;L) = \frac{zm\ell}{1 - z(1-m\ell)} , \tag{4.7}$$

which is the result (4.2), obtained for an unrestricted L.

From (4.4), we observe that for L=1, the number of records searched is geometrically distributed with parameter m. For unrestricted L, the relation (4.7) is observed to be the generating function of a random variable obeying the geometric distribution with parameter $m\ell$. Thus as ℓ increases, the distribution of $R(j,i;L)$ goes from geometric in m to geometric in $m\ell$.

It is of interest to observe how the expectation of $R(j,i;L)$ varies with L. This may be achieved by successive differentiation of (4.6).

By differentiating (4.6) with respect to z and evaluating the result at z=1, we obtain

$$E\{R(j,i;L)\} = \frac{1 - (1-\ell)^{L}}{m\ell} . \tag{4.8}$$

Equation (4.8) can be used to give some insight into how one might choose the length of the list. Let $R^*(j,i)$ be the number of records that the personnel managers "expect" to search in finding a person of category j at location i. Suppose also that m and ℓ are known for all locations and categories. Then, equating $R^*(j,i)$ to the right-hand-side of (4.8) we have

$$R^*(j,i) = \frac{1 - (1-\ell)^L}{m\ell} ,$$

or

$$L = \frac{\ell n(1-m\ell R^*(j,i))}{\ell n(1-\ell)} . \qquad (4.9)$$

Now, if $R^*(j,i)$ is on the order of $(m\ell)^{-1}$, then we see that L is very large. In this case the results of section B above are applicable.

On the other hand, if $R^*(j,i)$ is on the order of m^{-1}, then L is close to one, the minimum that L can take on in the context of the problem. If $R^*(j,i)$ is less than m^{-1}, then one would not "expect" to find any qualified personnel of category j in a search of $R^*(j,i)$ records, much less one who is currently at a specific location. Hence $m^{-1} \leq R^*(j,i) \leq (m\ell)^{-1}$ is the range of interest.

The next section is a discussion of the problem under restrictions on N, the number of records available to M*.

D. RESTRICTIONS ON THE NUMBER OF RECORDS AVAILABLE

Suppose now there are a limited number of records available to the personnel distribution center M* from which the selections are to be made. Then the introductory remarks in

section B give M(j), the number of records of category j, as being a binomial random variable with distribution parameter m(j).

Also equation (4.1) gives the generating function for G(j;k), the number of records searched to find the kth person of category j. However, (4.1) was derived assuming that the number of records available, N, is unrestricted. To determine the effect of restricting N, let $h(j;k|N)$ be the probability that there are at least k persons of category j in the N available records. That is,

$$h(j;k|N) = \Pr\{G(j;k) \leq N\}$$

$$= \sum_{n=k}^{N} \Pr\{G(j;k) = n\}. \tag{4.10}$$

$$= 1 - \sum_{n=0}^{k-1} \Pr \left\{\begin{array}{l}\text{exactly } n \text{ persons of} \\ \text{category } j \text{ in the } N \\ \text{records}\end{array}\right\}$$

$$= 1 - \sum_{n=0}^{k-1} \binom{N}{n} m^n (1-m)^{N-n}.$$

Let R(j,i;N) be the number of records searched in response to a demand from location i for a person of category j. Suppose that the search proceeds until the first person of category j who is currently available at location i is found or all N records have been searched. If all N records are searched without finding a person of category j, the demand is filed into a delay system to be serviced at a much later time, thus ensuring independent searches. If

upon searching the N records, one or more personnel of category j have found, but none are at location i, one of those found is transferred, ending the search.

Then we have

$$R(j,i;N) = \begin{cases} G(j,k) & \text{if the kth person found is the first to be located at } i \text{ and } G(j,k) \leq N, \\ N & \text{if at least one person of category } j \text{ was found, and none of those found were located at } i, \\ N + R'(j,i;N) & \text{if no personnel of category } j \text{ were found in the search of } N \text{ records,} \end{cases}$$

where $R'(j,i;N)$ is a random variable with the same distribution as $R(j,i;N)$. Then the generating function can be shown to be

$$R(j,i;N) = E\{z^{R(j,i;n)}\}$$

$$= \frac{1}{1-z^N(1-m)^N}\left(\frac{zm\ell}{1-z(1-m\ell)}\left(1 - \frac{m(1-z(1-m\ell))}{1-z(1-m)}\right)^N + z^N(1-m\ell)^N - z^N(1-m)^N\right).$$

It may be that the restriction on N is immaterial. Recall that $R(j,i)$ is the number of records searched to find the first person of category j who is currently at location i, where the selection list is unrestricted in length. From (4.2) it is straightforward to show that

$$E\{R(j,i)\} = (m\ell)^{-1}$$

and

$$V\{R(j,i)\} = (1-m\ell)(m\ell)^{-2}.$$

Now if N is greater than $E\{R(j,i)\}$ plus some number of standard deviations, one would "usually" expect to find a qualified replacement for the demand at the location from which the demand originated. That is, if

$$N \geq (m\ell)^{-1} + \gamma(1-m\ell)^{\frac{1}{2}}(m\ell)^{-1} \tag{4.11}$$

for all categories j and locations i, then the restrictions on the number of records available to M* is immaterial. For instance, if $R(j,i)$ is approximately normal, then

$$\Pr\{R(j,i) \geq N\} \leq \Pr\{R(j,i) \geq \frac{1}{m\ell} + \gamma\frac{1-m\ell}{(m\ell)^2}\}$$

$$= \Pr\{R(j,i) \geq n\}$$

and the latter term has values given by

γ	$\Pr\{R(j,i) \geq n\}$
1	.1587
2	.0540
3	.0013

For a given N satisfying (4.11) for a given γ, we see that the probability of not being able to locate an individual of a given category at a given location decreases rapidly with an increase in γ. That is, the movements between locations are not "usually" caused by the restrictions on N.

Note that γ is a decision variable. Given $m(j)$ and $\ell(j,i)$ for all categories j and locations i, one can find N as a function of γ by treating equation (4.11) as an equality. Doing this for all category/location combinations provides us with a set of N's. The element of this set with the greatest magnitude represents the number of records which must be made available to M* in order to ensure that the personnel distribution center is (to the degree represented by the magnitude of γ) unrestricted by N. Increasing γ implies increasing N*, the maximum of all the N generated. This reflects a greater requirement for M* to be able to locate a replacement of any category at any location.

The multiplier γ may be expressed as a function of location i and category j for all i and j. The multiplier would be larger for those locations and categories for which moves into or from the positions in question have high costs relative to other positions. For instance, in the three location problem of chapter III, we had that moves into and from location R were more expensive than other types of moves. In that case, we might set the multiplier γ to be a higher value than in other cases.

Making the multiplier γ a function of location i and category j has other applications. It may turn out that N*, the maximum of the calculated N, is too large for reality. The values N(j,i) which make up the set of N from which N* is the maximum may have only a few values which are "unreasonable". The γ for those particular locations/grades

might then be decreased, reflecting the decision to absorb the costs likely to be incurred due to the inability to find that particular combination of category and location in a reduced number of records available, and thereby reducing the value of N*.

The determination of N* may govern the size of the personnel distribution center M*. If N* is large, then many personnel may be required to maintain the selection process.

The next section formulates the problem for a restricted number of records available, N, and a restricted length selection list, L.

E. JOINT RESTRICTIONS ON THE LENGTH OF THE SELECTION LIST AND THE NUMBER OF RECORDS AVAILABLE

The most general case of the record search problem includes restrictions on both the number of records available to the personnel distribution center and the length of selection list that the personnel distribution center is required to establish prior to effecting a transfer.

It was noted previously that L, the length of the list, is a control variable, i.e., subject to institutional policy. The number of records available to the distribution center, denoted N, may also be a control variable. For instance, the value of N could be set by allowing only the next N people scheduled to rotate to be available. However, this may be artificial. A more realistic situation is now described.

Suppose that all tour lengths for personnel of category j, denoted T_j, are fixed. Suppose also that N_j denotes the number of personnel of category j in the system. Suppose now that the institution sets an "availability period" which gives the period just prior to the scheduled rotation date during which the individual concerned is available to the personnel distribution center. Denote the availability period for a person of category j as a_j. Then the number of people of category j available to the distribution center, denoted A_j, is, on the "average"

$$A_j = \frac{a_j}{T_j} N_j .$$

Thus, the number of availabilities is seen to be a function of two policy decisions, the setting of tour lengths and the setting of the availability periods.

Prior to embarking on the mathematical description of the model under joint restrictions on N and L, we discuss the sequential nature of the record search problem. Essentially, the personnel distribution center searches the available records in response to a demand until one of four possible situations is encountered:

(1) During the search, a person of category j at location i is found. The first person so discovered is transferred and the cost of searching the records is incurred.

(2) The search terminates with L people in category j, none of whom are currently at location i. One of the qualified personnel is transferred, and both a transfer cost and a cost of searching is incurred.

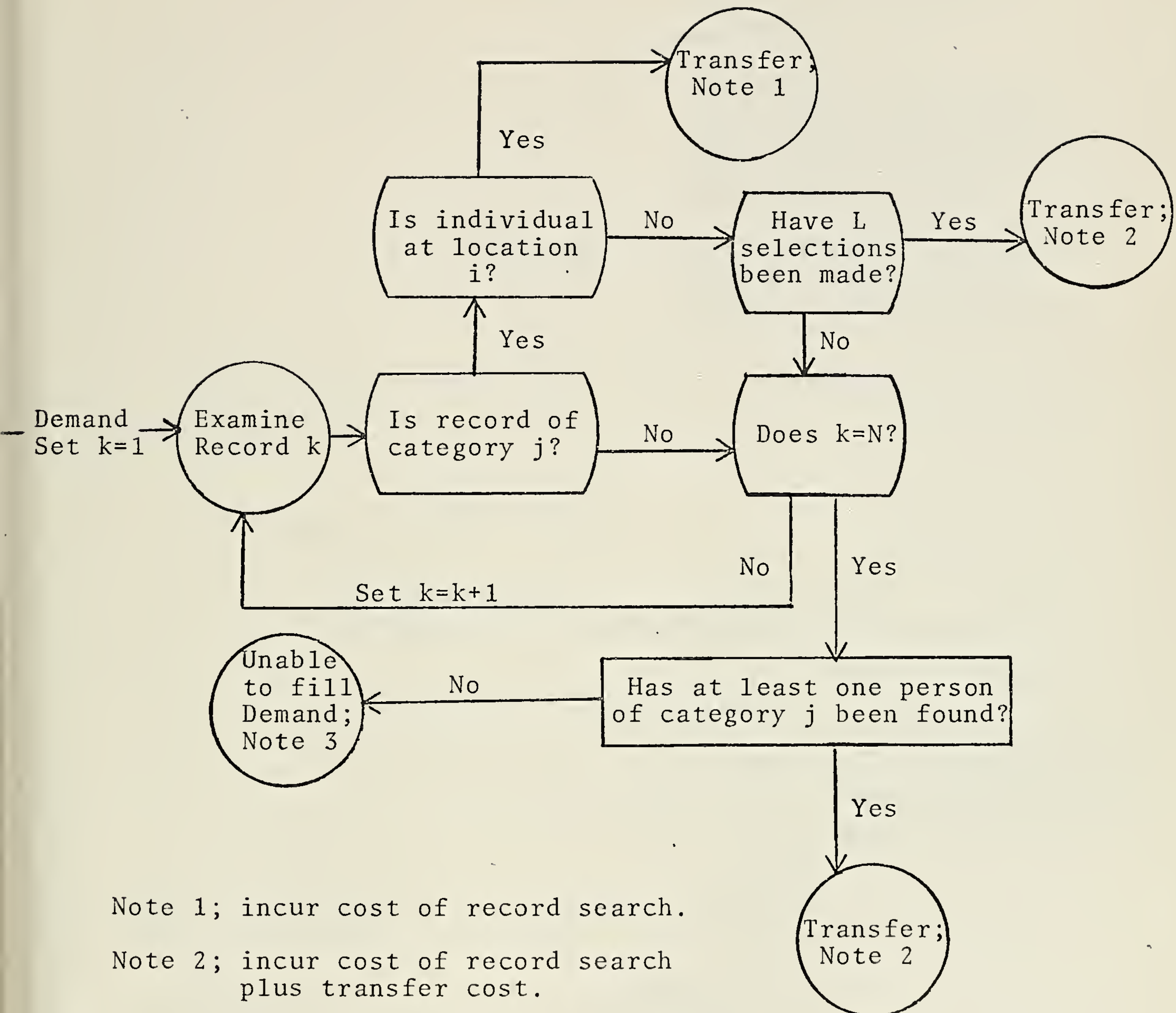
(3) The search terminates after searching N records, at least one of whom is of category j. Again, a transfer cost is incurred as well as a record search cost. Note that the number of selected individuals in this case is strictly less than L, there will be less flexibility in the decision of whom to transfer, so one might expect the transfer cost to be greater.

(4) The search terminates after searching N records, none of whom is of category j. The cost of searching N records is incurred. Also, a "penalty" cost of not being able to locate a suitable replacement in response to a demand may be incurred. In any case, some time later the distribution center will again search for such a replacement, incurring additional costs.

The sequential nature of the problem under joint restrictions on N and L is illustrated in figure 4.1.

Suppose now that the personnel distribution center files an unfilled demand into a delay system, from which the demand will emerge at a later time. Suppose the delay is so great that the personnel available to the distribution center during the first search are completely different from those available during the second, or subsequent, search. Thus, the searches are independent. Let R(j,i;N,L) be the number of records searched to satisfy a demand from location i for a person of category j under restrictions on N and L. Then

$$R(j,i;N,L) = G(j;k) \quad \text{if the kth person of category j is the first to be located at i, } k=1,\ldots,L, \text{ and } G(j;k) \leq N,$$



Note 1; incur cost of record search.

Note 2; incur cost of record search plus transfer cost.

Note 3; incur cost of record search plus penalty cost.

Figure 4.1. The Record Search Problem with Restrictions on N and L, Showing Response to a Demand From Location i for a Person of Category j.

$$R(j,i;N,L) = G(j;L)$$ if the Lth person of category j to be found is not at location i and $G(j;L) \leq N$,

$$= N$$ if L persons of category j were not found, at least one qualified person was found, and none of those found were at i,

$$= N + R'(j,i;N,L)$$ if no individual of category j was found in a search of the N records.

The generating function for $R(j,i;N,L)$ can now be found. We obtain

$$\hat{R}(j,i;N,L) = E\{z^{R(j,i;N,L)}\}$$

$$= \sum_{k=1}^{L} E\{z^{G(j;k)}\}h(j;k|N)(1-\ell)^{k-1}\ell$$

$$+ E\{z^{G(j;L)}\}h(j;L|N)(1-\ell)^{L}$$

$$+ z^{N}\{h(j,1|N)-h(j,L|N)\}\Pr\{\text{none of those found were at location } i\}$$

$$+ z^{N}E\{z^{R'(j,i;N,L)}\}(1-h(j;1|N)).$$

After some rather messy algebra, we obtain

$$\hat{R}(j,i;N,L)\,(1-z^{N}(1-m)^{N}) = \frac{zm\ell}{1-z(1-m\ell)}$$

$$+ \left(\frac{zm(1-\ell)}{1-z(1-m)}\right)^{L}\left(1 - \frac{zm\ell}{1-z(1-m\ell)}\right)\sum_{n=L}^{N}\binom{N}{n} m^{n}(1-m)^{N-n}$$

$$- \frac{zm\ell}{1-z(1-m\ell)}\sum_{n=0}^{L-1}\binom{N}{n}\left(\frac{zm^{2}(1-\ell)}{1-z(1-m)}\right)^{n}(1-m)^{N-n}$$

$$- z^N(1-m)^N + z^N \sum_{n=0}^{L-1} \binom{N}{n} m(1-\ell)^n(1-m)^{N-n}.$$

Thus the general problem of restricted N and L does not seem to possess sufficient tractability to proceed analytically. The effect of the restrictions on other aspects of the distribution problem remains a problem for future reconciliation. Although there is not much insight to be gained from the mathematics, one does gain appreciation of the difficulties to be encountered in the analysis of the operation of the personnel distribution center.

F. EXTENSIONS AND CONCLUSIONS

In this chapter, we found that increasing the length of the selection list and/or restricting the number of records available to the personnel distribution center affects the numbers of records that must be searched prior to making a decision to effect a transfer. This in turn affects the amount of time spent in searching in response to each individual demand.

Again, the results of chapter II might affect the search procedure. For instance, instead of searching for the first qualified person to be found who is currently at the location from which the demand originated, one might only search until finding the first qualified person whose transfer to fill a demand is among those specified as being among the optimal set of transfers under the provisions of chapter II.

Alternatively, one might set bounds on the number of candidates to be selected prior to effecting a transfer. That is, there may be a value $\underline{L}$ denoting the number of candidates to be selected prior to effecting a transfer, provided no candidate is a "no-cost" transfer, i.e. is at the location from which the demand originated. Once $\underline{L}$ candidates have been selected, then the candidates are examined to determine if there are any whose transfer would be in the optimal set provided by chapter II. If so, then one of those is transferred. If not, the search continues until either a candidate whose transfer would be in the optimal set is selected or $\overline{L}$, a maximum number of candidates, is achieved. In the later case, a "non-optimal" transfer is effected. The analysis of such a procedure would be much more complex than that presented herein. Since we found the results of this chapter to be difficult to interpret, and such extensions would be more so, we did not pursue the extensions.

Another extension, also not pursued for the same reason, would be the examination of the effect of allowing degrees of qualification to enter the problem. In this extension, personnel might be classified as qualified, over-qualified, marginally-qualified, and not qualified. The inability to select an individual that is qualified might result in the assignment of an over-or marginally-qualified individual. Such a problem would be faced with the determination of "penalty" costs for using an over-or marginally-qualified

person in a position as well as the transfer costs associated. Here again, the concepts of a lower and upper bound on the length of the selection list might be useful.

V. CONCLUSION

In this thesis, we have developed three analytical models which serve to describe rotations in multi-location, multi-grade personnel systems. The models, although basically independent, have been shown to interact on a macroscopic level.

The basic balance equations which we developed in chapter II are the first known analytical relationships describing the complex interactions between billet structure, rotation structure and promotion structure in a large institution. Under the assumptions of chapter II, the structures are seen to be heavily dependent on each other. Thus we were able to show that the specification of the billet structure and the rotation structure completely determined the promotion structure. This high degree of dependence may explain the problems encountered in a personnel system where the three structures are specified independently.

In the development of the sensitivity analysis on the transportation problem, we found that changes in one element of the basic structures had complex ramifications on the constraint vector of the basic transportation problem. Thus we developed the "secondary" analysis given in section II.I, where the feasibility of the primal problem was considered explicitly in the computation of the shadow prices.

In chapter III, we developed a model of the personnel selection process within a large institution, where the

selections are made to determine which individual transfer is to be made in response to a demand for a person of a specified qualification. The model, which we called the Ehrenfest Decision Model, was developed for institutions with two and three locations. The model serves to parameterize the personnel selection process. The results specify that lengthening the selection list, i.e. increasing the number of qualified personnel to be found prior to making a decision, serves to decrease the numbers of interlocation transfers which occur.

In chapter IV, we developed a model which describes the problems encountered by the personnel manager when the selection list is lengthened. The results of this model indicate that one would not desire to force the selection list to be arbitrarily long, since such action implies that the number of records which one expects to be searched in order to specify the list may become very large. The large number of records would in turn imply a long search time and thus an increasing degree of unresponsiveness to arriving demands.

In the conclusions to both chapter III and IV, the influence of the results of chapter II on the models of those chapters was discussed.

Finally, the pursuit of better and better understanding of the complex interactions between components of the personnel systems of large institutions has been our goal. The complexity is of increasing concern as more and more institutions enter the family of institutions with multi-location,

multi-grade personnel systems. Also, those institutions already in the "family" are interested in becoming more and more efficient in their utilization of personnel. Diseconomies caused by failure to understand these interactions are of concern to both institutional directors and their constituents, both public and private.

APPENDIX A

APL programs used in chapter II.

a) PROMATB

The function PROMATB performs the calculations of example 1. That is, it takes as input the billet structure B, the rotation rates R and the vector of withdrawal rates w. The unique promotion matrix Q which satisfies the basic balance equations and assumption A2;4 is then calculated. The function uses the function QCALC as a subroutine.

```
      ∇ PROMATB;M;RO;IQINV;G;K;V;RQ
[1]     'ARE THERE ANY CHANGES TO BILLET STRUCTURE?'
[2]     →(0=⎕)ρL3
[3]    L2:'SPECIFY LOCATION, PAY GRADE, NEW BILLET SIZE'
[4]     B[V[1];V[2]]←V[3],0ρ,V←⎕
[5]     'ANY MORE CHANGES?'
[6]     →(1=⎕)ρL2
[7]    L3:RQ←BxR
[8]     QCALC
[9]     IQINV←(⌹((G,G)ρ1,Gρ0)-Q)[1;]
[10]    'CONTINUE?'
[11]    →(1=⎕)ρL2
      ∇
```

The effect of changes in the billet structure on the promotion scheme refer, for instance, to examples 2 and 3, can be observed through application of PROMATB. Line 1 asks the user whether or not there are to be any changes in the billet structure. As with other programs described in this section, an answer of 1 means affirmative and an answer of 0 means a negative reply. Line 3 tells the user who has indicated that changes are to be made to specify the changes.

With this information, a new Q is calculated, assuming that all other factors remain fixed.

2) RELOCATE

The function RELOCATE calculates the new billet structure following a change in a particular element of B under the conditions of example 4. Recall that in example 4 of chapter II, the billets in each grade were fixed, and thus the total population of the institution was fixed. The change in billets at one location for a particular grade was to be absorbed proportionately throughout the rest of the system.

Following application of RELOCATE, the new billet structure thus obtained is used as an input to PROMATB in order to generate a new promotion matrix Q.

```
        ∇  RELOCATE;G;L;T;A
[1]        'WHAT LOCATION, PAY GRADE AND NEW BILLET SIZE?'
[2]        G←T[2],0ρ,L←T[1],0ρ,T←⎕
[3]        B[;G]←B[;G]x(A-T[3])÷(A←(+/B)[G])-B[L;G]
[4]        B[L;G]←T[3]
        ∇
```

3) ROTCOMP

The function ROTCOMP calculates the new rotation rates for a particular grade when the rotation rate for personnel of that grade at a specified location is changed. The rates for personnel of that grade at other location shift proportionately. There is no effect on the other parameters of the system. This was used in the calculations of example 8.

```
     ∇    ROTCOMP;V;G;L;A;C;BETA
[1]       'SPECIFY LOCATION GRADE AND NEW ROTATION RATE'
[2]       L←V[1],0ρ,G←V[2],0ρ,V←⎕
[3]       A←÷R[L;G]÷V[3]-R[L;G]
[4]       BETA←-AxC÷(+/BxR)[G]-C←B[L;G]xR[L;G]
[5]       R[;G]←R[;G]x1+BETA
[6]       R[L;G]←V[3]
     ∇
```

4) PROMATR

As opposed to PROMATB, which takes the billet structure an an input, PROMATR takes the requirements of several periods and calculates the promotion schemes for each of the periods. It is implicit in PROMATR that the tour lengths specified and the requirements are compatible, i.e. r(i;k;n) = r(i;k;n-T(i;k)) for all locations i, grades k and time periods n. Thus PROMATR does the calculations under assumptions of weak stationarity.

```
     ∇    PROMATR;V;IN;G;N;RQ;IQINV
[1]       IN←1
[2]       N←V[1],0ρ,G←V[3],0ρ,V←ρRQN
[3]       'ARE THERE ANY CHANGES TO SPECIFIED REQUIREMENTS?'
[4]       →(0=⎕)ρL2
[5]      L1:'SPECIFY PERIOD,LOCATION,PAY GRADE AND NEW REQUIREMEN
[6]       RQN[V[1];V[2];V[3]]←V[4],0ρ,V←⎕
[7]       'ANY MORE CHANGES?'
[8]       →(1=⎕)ρL1
[9]      L2:RQ←RQN[IN;;]
[10]      QCALC
[11]      EXN[IN;;]←-RQ-A←RQ+.xQ
[12]      QN[IN;;]←Q
[13]      RON[IN;;]←+/RQ+.xW
[14]      →(N≥IN←IN+1)ρL2
     ∇
```

5) STATPRO

STATPRO develops a stationary promotion scheme for systems which have only weak stationarity. The billet structure supported by a rotation scheme and requirements as used in PROMATR

is used in PROMATB to generate the promotion scheme. The disparities between availabilities and requirements arising due to the forcing of a stationary promotion scheme onto a non-stationary structure are provided as an output.

```
        ∇  STATPRO;IN;EN;N;V
[1]        PROMATB
[2]        IN←1,0ρ,N←V[1],0ρ,V←ρRQN
[3]       L1:RQ←RQN[IN;;]
[4]        EXN[IN;;]←-RQ-A←RQ+.xW
[5]        →(N≥IN←IN+1)ρL1
        ∇
```

6) QCALC

QCALC is used as a subroutine in PROMATB and PROMATR, performing the actual calculations leading to the determination of the promotion matrix Q.

```
        ∇  QCALC;K;RO;G
[1]        G←ρIQINV←(+/RQ)÷RO←+/RQ+.xW
[2]        Q←(G,G)ρ0,0ρ,K←2
[3]        Q[1;2]←1-(Q[1;1]+W[1]),0ρ,Q[1;1]←1-÷IQINV[1]
[4]       L1:Q[K;K+1]←1-(Q[K;K]+W[K]),0ρ,Q[K;K]←1-IQINV
                          [K-1]xQ[K-1;K]÷IQINV[K]
[5]        →((G-1)>K←K+1)ρL1
[6]        Q[G;G]←1-W[G]
        ∇
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